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SOME OBSERVATIONS ABOUT THE RMS RING FOR DELAYED SYSTEMS

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Abstract: This paper questions the validity of the definition and some properties of the $\mathbf{R}_{MS}$ ring traditionally utilized for the description of time-delay systems. The original description of the ring is faced with findings obtained while dealing with this ring. In the light of these observations, it seems that a revisited definition of the ring ought to be formulated, and thus a new possible conception is presented in the contribution. It is also shown in the paper that the $\mathbf{R}_{MS}$ ring proposed in the paper is not a unique factorization domain and thus it is not a principal ideal domain. The extended Euclidean algorithm is attempted to be performed for the ring to prove that it is a Bézout domain, which induces the question of existence of the coprime factorization for each pair of elements of the ring. These two problems are discussed; however, they remain partially unsolved.

Keywords: Time-delay systems, algebraic approach, $\mathbf{R}_{MS}$ ring.

1 INTRODUCTION

Algebraic structures proved to be suitable and effective tools for system dynamics description and control system design and thus modern control theory has adopted algebraic approaches and parlance for decades.

These approaches are based on fractional description of systems. From the historical point of view, the polynomial ring representation played the primarily part; especially due to the natural correspondence between the transform and the time-domain description for discrete-time systems (Kalman *et al.* 1969), (Kučera 1979). A more general description of systems brought the introduction of fractional fields of appropriate rings (Desoer *et al.* 1980), (Vidyasagar 1985), (Kučera 1993). The transfer function of a system is then an element of a field of fractions over an appropriate ring. This description is suitable particularly when it is desirable to obtain certain control performance and a controller structure via algebraic controller design dealing with linear Diophantine equations (Hautus 1976), (Callier and Desoer 1982), (Kučera 1983), (Vidyasagar 1985), and it can be extended to cover continuous-time systems.

One of such rings for continuous-time systems is a ring of stable and proper rational functions, $\mathbf{R}_{PS}$ (Kučera 1993), (Prokop and Corriou 1997), (Dlapa and Prokop 2008). An element of this ring is expressed as a ratio of two polynomials where the denominator polynomial is Hurwitz stable (i.e. free of roots located in the complex right-half plane including imaginary axis) and, moreover, the ratio is proper (i.e. the degree of the numerator is less or equal to the denominator). Hence, the element of $\mathbf{R}_{PS}$ is analytic for $\mathrm{Re}(s) \geq 0$ including $s = \infty$. Any $A(s) \in \mathbf{R}_{PS}$ divides $B(s) \in \mathbf{R}_{PS}$ iff all unstable zeros (including $s = \infty$) of $A(s)$ are those of $B(s)$.

The process of transcription of the transfer function into the ratio of two coprime elements of a ring is called a coprime factorization. An example of this factorization for $\mathbf{R}_{PS}$ follows. Assume a plant transfer function

$$G(s) = \frac{1}{s-1} \tag{1}$$

as a ratio of polynomials obtained directly from the Laplace transform of the plant differential equation. To acquire the transfer function in the $\mathbf{R}_{PS}$ ring, use e.g. the following factorization

$$G(s) = \frac{\frac{1}{s+a}}{\frac{s-1}{s+a}} \qquad (2)$$

The common denominator, $s+a$, must be of degree one; otherwise, there would either exist a common unstable pole ($s = \infty$) and thus the ratio would not be coprime, or the numerator or denominator of the transfer function would not be from $\mathbf{R}_{PS}$ (proper). A parameter a is a positive real number ensuring the stability of the common denominator.

Algebraic controller design in this ring via the solution of a linear Diophantine equation ensures the internal stability of the feedback system and leads to a proper controller.

The presence of delays, in input-output relation or as state delays, however brings a convenience to define another ring instead of $\mathbf{R}_{PS}$ since this ring requires a rational approximation of exponential terms. Obviously, such conversion brings about certain loss of information of the system behavior. This is evident particularly in case anisochronic (i.e. containing delayed states) systems which belong to the set of infinite-dimensional linear systems. One of the first attempts to utilize algebraic theory to infinite-dimensional linear systems was made by Kamen (1975) where an operator theory was presented. Almost in the same time, Morse (1976) and Sonntag (1976) introduced some general rings for delayed systems dealing with polynomials in two variables R[s, z] over real numbers where $z = \exp(s)$. In Kamen *et al*. (1986), the ring Θ generated by the entire functions

$$\frac{1-\exp(-s+\sigma)}{s-\sigma},\ \sigma \in C \qquad (3)$$

and their derivatives is studied. Behavioral approach, as it was introduced for dynamical systems in (Willems 1989), for linear time-invariant delay-differential systems is presented by Gluessing-Lueerssen (1997). In contrast to above mentioned works, she considered systems in the behavioral point of view instead of systems over rings.

The definition of the $\mathbf{R}_{MS}$ ring, i.e the ring of stable and proper retarded quasipolynomial meromorphic functions, is introduced in (Zítek and Kučera 2003). These authors took into account the fact that two variables, z and s, are not independent, thus, this algebraic approach is one-dimensional. The Laplace transform of linear delayed (including anisochronic) systems results in transfer functions those are ratios of so-called quasipolynomials, see e.g. (El'sgol'ts and Norkin, 1973). Similarly to $\mathbf{R}_{PS}$, these transfer functions can be factorized in the form of ratios of two elements from a special ring, $\mathbf{R}_{MS}$. An element of the $\mathbf{R}_{MS}$ ring as a ratio of quasipolynomials is, however, no more rational and it belongs to the more general set of meromorphic functions. It should be also noted that $\mathbf{R}_{PS} \subseteq \mathbf{R}_{MS}$.

Nevertheless, a practical dealing with this ring, as it is defined in (Zítek and Kučera 2003), revealed some discrepancies which ought to be reviewed and discussed. Therefore this contribution focuses some apparent problems of the $\mathbf{R}_{MS}$ ring original definition and some its algebraic properties are presented here, and the paper also suggests an alternative new conception of the ring. We observed that the original $\mathbf{R}_{MS}$ is not a ring indeed. Moreover, we revised the stability properties of an element from $\mathbf{R}_{MS}$ and divisibility conditions. It is also presented that the concept should not be restricted to retarded systems only, but neutral systems ought to be taken in account as well. It is shown that the new conception of the ring is neither a unique factorization domain nor a principal ideal domain. The Diophantine equation and its solution using the extended Euclidean algorithm in the ring are suggested. However, the task of the general existence of the solution of this equation and that of a coprime factorization remain unsolved. Particular examples rather then rigorous mathematical derivations and proofs are given to demonstrate the problems and properties.

2 THE ORIGINAL DEFINITION OF THE RMS RING

The important definitions concerning the $\mathbf{R}_{MS}$ ring presented in (Zítek and Kučera 2003) follow.

Definition 1 (Retarded quasipolynomial): A quasipolynomial of the generic form

$$M(s) = s^n + \sum_{i=0}^{n-1}\sum_{j=1}^{h_i} m_{ij} s^i \exp(-\eta_{ij} s),\ \eta_{ij} \geq 0 \qquad (4)$$

i.e. its highest s-power s^n is delayless.

Definition 2 (Stability of retarded quasipolynomial): A retarded quasipolynomial $M(s)$ is said to be stable if it does not have any finite zero s_0 such that $\mathrm{Re}(s_0) \geq 0$.

Definition 3 (RQ meromorphic function): A ratio of quasipolynomials $N(s)/M(s)$ is said to be retarded-quasipolynomial (RQ) meromorphic function if

1. $M(s)$ is retarded quasipolynomial as in (4) and
2. $N(s)$ can be factorized as $N(s) = \tilde{N}(s)\exp(-\tau s)$, where $\tau > 0$ and $\tilde{N}(s)$ is a retarded quasipolynomial and
3. the fraction is proper, i.e. it holds for the highest s-power s^m in $N(s)$, that $m \leq n$.

Definition 4 (RQ asymptotic stability): An RQ meromorphic function is said to be stable if it is analytic in the closed right half *s*-plane, i.e. if its denominator retarded quasipolynomial $M(s)$ is stable.

Stability of a retarded quasipolynomial can be found e.g. in (Zítek 1997): A retarded quasipolynomial $M(s)$ is stable iff $M(0)=0$ and

$$\underset{s=\omega j,\omega\in[0,\infty]}{\Delta \arg} M(s)=\frac{n\pi}{2} \tag{5}$$

where again *n* is the highest *s*-power of $M(s)$.

Then the ring of stable RQ meromorphic functions is denoted by $\mathbf{R}_{MS.}$ In the controller design, both the plant and controllers are to be considered as ratios of two coprime elements from $\mathbf{R}_{MS}$. The added denominators can be either stable polynomials or quasipolynomials.

The division condition for $\mathbf{R}_{MS}$ is as the same as for $\mathbf{R}_{PS}$, i.e. $A(s)\in \mathbf{R}_{MS}$ divides $B(s)\in \mathbf{R}_{MS}$ iff all unstable zeros (including $s = \infty$) of $A(s)$ are those of $B(s)$.

An example presented in the discussed literature follows. Consider a plant described by the transfer function

$$G(s)=\frac{\exp(-\tau s)}{s} \tag{6}$$

The following coprime fraction can be used

$$G(s)=\frac{B(s)}{A(s)}=\frac{\dfrac{\exp(-\tau s)}{Ts+\exp(-\vartheta s)}}{\dfrac{s}{Ts+\exp(-\vartheta s)}} \tag{7}$$

The common quasipolynomial denominator is stable iff

$$\frac{\vartheta}{T}<\frac{\pi}{2},\ T>0,\ \vartheta\geq 0 \tag{8}$$

see e.g. in (Górecki *et a*l. 1989).

I should be noted that the authors restricted the utilization of the ring onto time delay systems containing lumped delays. However, as it is shown in (Zítek and Víteček 1999), models of systems with distributed delays expressed by convolutions can be rewritten to models with lumped delays (i.e. exponentials in the transformation).

3 OBSERVATIONS ABOUT THE ORIGINAL DEFINITION

The $\mathbf{R}_{MS}$ ring together with the comprime factorization can be used in the process of algebraic design of feedback controllers via the solution of the Diophantine (Bézout) equation, see e.g. (Zítek and Kučera 2003), (Pekař and Prokop, 2007). Dealing with $\mathbf{R}_{MS}$, however, arises some questions about the validity of the original definition. The following observations and notes should be perceived as "things-to-thought". The authors of this paper set the great store by the work of the authors of the original definition, and their intention is not to fully vitiate the original conception of $\mathbf{R}_{MS}$.

3.1 *Remarks on Definition 3*

It is stated in the paragraph 2 of the definition of an RQ meromorphic function that the numerator of a function can be factorized as a product of a delayless retarded quasipolynomial and an exponential term satisfying $\tau > 0$. However, it does not always hold as it is obvious from the example (7), since $A(s)$ does not contain explicitly a delay term. We assume that the restriction to the exponential should be $\tau \geq 0$ rather than $\tau > 0$.

3.2 *Remarks on Definition 4*

An RQ meromorphic function is asymptotically stable iff it is analytic in the closed right half plane. For a ratio of polynomials, this condition is equivalent to the statement that the denominator is stable polynomial, i.e. the fraction has poles located in the open left *s*-plane only.

A ratio of retarded quasipolynomials (or a retarded quasipolynomial and a polynomial), however, has a rather different properties. Since a retarded quasipolynomial owns an infinite number of its zeros, there can exists a fraction of retarded quasipolynomials with some common unstable roots of both elements without possibility to cancel any factor. This implies that there can exist unstable roots of the denominator so that the whole fraction is analytic (holomorphic) in the complex right half plane. In this case, the singularity is not called a pole but *a removable singularity*.

The necessity to extend the stability conditions as suggested above is motivated also by some results obtained from algebraic controller design in $\mathbf{R}_{MS}$. The following example demonstrates it.

Let the plant transfer function reads

$$G(s)=\frac{K\exp(-\tau s)}{Ts+\exp(-\vartheta s)}=\frac{\dfrac{K\exp(-\tau s)}{s+m_0}}{\dfrac{Ts+\exp(-\vartheta s)}{s+m_0}}=\frac{B(s)}{A(s)} \tag{9}$$

where $K, T, \tau, \vartheta, m_0$ are positive real parameters. Consider the controller

$$G_R(s)=\frac{Q(s)}{P(s)} \tag{10}$$

A particular solution of the stabilizing Diophantine equation, see e.g. (Kučera 1993),

$$A(s)P(s)+B(s)Q(s)=1 \tag{11}$$

is

$$Q_P(s)=1,\ P_P(s)=\frac{s+m_0-K\exp(-\tau s)}{Ts+\exp(-\vartheta s)} \tag{12}$$

The set of all internally stabilizing controllers is then given by the Youla-Kučera parameterization

$$\begin{aligned}P(s)&=P_P(s)-B(s)T(s)\neq 0\\ Q(s)&=Q_P(s)+A(s)T(s)\end{aligned} \tag{13}$$

where $T(s)$ is a free element of the ring.

The parameterization of the solution of the Diophantine equation enables to fulfill other control performance conditions besides internal stability (Vidyasagar 1985, Kučera 1993). Consider a step reference signal

$$W(s)=\frac{1}{s}=\frac{\frac{1}{s+m_0}}{\frac{s}{s+m_0}}=\frac{H_W(s)}{F_W(s)} \tag{14}$$

Analogously to the $\mathbf{R}_{PS}$ ring, the requirement of the asymptotical zero control error is conditioned by that the image of the control error is an element of the ring, i.e.

$$E(s)=A(s)P(s)\frac{H_W(s)}{F_W(s)}\in \mathbf{R}_{MS} \tag{15}$$

in other words, $F_W(s)$ must divide $P(s)$. Hence, taking

$$T(s)=\frac{(m_0/K-1)(s+m_0)}{Ts+\exp(-\vartheta s)} \tag{16}$$

it is obtained from (12)

$$P(s)=\frac{s+m_0[1-\exp(-\tau s)]}{Ts+\exp(-\vartheta s)} \tag{17}$$

Indeed, the only unstable zero of $F_W(s)$ is $s_0=0$ which is a zero of $P(s)$ as well. Thus, $F_W(s)$ divides $P(s)$ and the quotient

$$\frac{P(s)}{F_W(s)}=\frac{(s+m_0[1-\exp(-\tau s)])(s+m_0)}{s[Ts+\exp(-\vartheta s)]} \tag{18}$$

is from $\mathbf{R}_{MS}$. It means that a removable singularity $s_0=0$ is allowed to be a root of the denominator of the element from the $\mathbf{R}_{MS}$ ring here. If all singularities are removable, the function becomes *holomorphic (*or even *entire).*

3.3 *New divisibility condition*

The example presented in the previous subsection arises the question of validity of the division condition for $\mathbf{R}_{MS}$. Term (18) is obviously proper and analytic in the open complex right-half plane including infinity, thus, the condition of divisibility presented above holds for this example.

Consider now, however, another example. Let

$$X(s)=\frac{s}{s+m_0},\ Y(s)=\frac{1-\exp(-s)}{s} \tag{19}$$

and the ratio

$$\frac{Y(s)}{X(s)}=\frac{[1-\exp(-s)](s+m_0)}{s^2} \tag{20}$$

be given. The previously presented divisibility condition holds; however, the ratio (20) is not analytic at $s_0=0$. This example suggests a new possible condition of the divisibility in $\mathbf{R}_{MS}$.:

$A(s)\in\mathbf{R}_{MS}$ divides $B(s)\in\mathbf{R}_{MS}$ iff all unstable zeros (including $s=\infty$) of $A(s)$ are those of $B(s)$, *and* all unstable poles (including $s=\infty$) of $B(s)$ are those of $A(s)$.

3.4 *Ring conditions*

The authors of the original definition of $\mathbf{R}_{MS}$ delimitated the utilization of the ring onto *retarded* delayed systems only. This subsection suggests extending it onto *neutral* systems as well, with some limitations.

Look at ring properties of $\mathbf{R}_{MS}$. The existence of the additive and multiplicative identity and the additive inverse is obvious. The associativity of addition and multiplication and the commutativity of addition can be easily deduced from properties of a quasipolynomial. The only problem brings the requirement of the closure axiom for addition. A simple example

$$X(s)=\frac{s+2}{s+m_0},\ Y(s)=\frac{s+1}{s+m_0}\exp(-s) \tag{21}$$

demonstrates that the summation

$$X(s)+Y(s)=\frac{[1+\exp(-s)]s+2+\exp(-s)}{s+m_0} \tag{22}$$

is *not* an element of $\mathbf{R}_{MS}$ due to non-retarded quasipolynomial in the denominator, thus $\mathbf{R}_{MS}$ cannot be a ring. However, due to usual strict properness of a plant, the final controller obtained by algebraic controller design is free of non-retarded quasipolynomial in its transfer function.

Systems which dynamics is expressed by a quasipolynomial of the form

$$\begin{aligned}&M(s)=s^n+\sum_{i=0}^{n}\sum_{j=1}^{h_i}m_{ij}s^i\exp(-\eta_{ij}s),\ \eta_{ij}\geq 0,\\ &\sum_{j=1}^{h_n}m_{nj}\exp(-\eta_{nj}s)\neq \text{const.}\end{aligned} \tag{23}$$

are called *neutral* (Hale and Lunel 1993). Thus, quasipolynomials of the form (22) are called neutral as well. The stability condition for such quasipolynomials was investigated by Zítek and Vyhlídal (2008):

A quasipolynomial of the form (23) is (even *strongly*) stable if

1. $\sum_{j=1}^{h_n} \left| m_{nj} \right| < 1$ and
2. $M(0) > 0$ and
3. $\frac{n\pi}{2} - \Phi \leq \underset{s=\omega j, \omega\in[0,\infty]}{\Delta \arg} M(s) \leq \frac{n\pi}{2} + \Phi$

where $\Phi = \arcsin\left(\sum_{j=1}^{h_n} \left| m_{nj} \right| \right)$. The most significant difference in stability properties between retarded and neutral quasipolynomials is (besides sensitivity to delay changes) in the number of unstable poles. Whereas an unstable retarded quasipolynomial can have only a *finite* number of unstable roots, a neutral one has an *infinite* number of unstable roots. This property brings problems while maintaining the coprime factorization as it is demonstrated below in Section 4.

To be a ring, $\mathbf{R}_{MS}$ has to adopt a neutral quasipolynomial in the numerator of an element of the ring. However, in the light of this fact, notion *retarded* would be omitted. There is also no obstacle to put a neutral quasipolynomial in the denominator if the whole term satisfies the properties of stability and properness.

4 SOME PROPERTIES OF THE NEW CONCEPTION

4.1 *Units of the ring*

Generally, a *unit* of a ring is its element, the multiplicative inverse of which is in the ring again. One can easily deduced that a unit of $\mathbf{R}_{MS}$ is an element of zero relative order having the same (all) unstable roots of the quasipolynomial numerator and denominator.

Elements which arise by multiplication by a unit are said to be *associated*.

4.2 *Unique factorization domain and principal ideal domain*

The $\mathbf{R}_{MS}$ ring characterized in Section 3 is obviously commutative. Quasipolynomial properties also determine that it is an integral domain. The question is whether the ring is a *unique factorization domain* (UFD).

To prove that the $\mathbf{R}_{MS}$ ring is not a UFD, it is sufficient to show that there exists a non-zero non-unit element of the ring which cannot be written as a finite product of irreducible elements.

Consider the following element of the ring

$$\frac{1-\exp(-\tau s)}{s} \tag{24}$$

In spite of the fact that this expression is more than rare to be useful for system description or controller design, it is an element of the ring anyway.

Non-zero zeros of (24) are

$$s_{k,k+1} = \pm \frac{2k\pi}{\tau} \text{ j}, \; k \in \mathbf{N} \tag{25}$$

Define polynomials

$$P_k(s) = (s - s_k)(s - s_{k+1}) \tag{26}$$

Then the factorization

$$\begin{aligned} &\frac{1-\exp(-\tau s)}{s} = \\ &= \frac{[1-\exp(-\tau s)](s+m_0)^2}{sP_1(s)} \frac{P_1(s)}{(s+m_0)^2} = \\ &= \frac{[1-\exp(-\tau s)](s+m_0)^4}{sP_1(s)P_3(s)} \frac{P_1(s)P_3(s)}{(s+m_0)^4} = \\ &\ldots \end{aligned} \tag{27}$$

where $m_0 > 0$ is real, is infinite and thus the $\mathbf{R}_{MS}$ ring is not a UFD, and none of left-hand factors in (27) is irreducible and none of all factors is a unit.

It is well known fact that *principal ideal domain* (PID) $\Rightarrow$ UFD, see e.g. (Barile *et al*. 2009). Hence, $\mathbf{R}_{MS}$ is not a PID, i.e. there exists an ideal which cannot be generated by a single element of the ring.

4.3 *Bézout domain and the extended Euclidean algorithm*

The goal of this subsection is the endeavor to show that the $\mathbf{R}_{MS}$ ring is a *Bézout domain*, i.e. every *finitely* generated ideal is principal. In other words, whether any two elements $A(s), B(s) \in \mathbf{R}_{MS}$ have a greatest common divisor (GCD), $D(s)$, that is a linear combination of them

$$A(s)X(s) + B(s)Y(s) = D(s) \tag{28}$$

If the ring is a Bézout domain, then there exists a coprime factorization of $A(s), B(s)$ satisfying Bézout identity (11), see e.g. (Doyle *et al*., 1990), which is important for controller design mentioned in Subsection 3.2.

Hence, if we can define the *extended Euclidean algorithm* for $\mathbf{R}_{MS}$, the ring is a Bézout domain (Rosický, 1994). It is assumed that a reader is already familiar with Euclidean algorithm for integers.

Generally, the iterative method of the extended Euclidean algorithm for given *A* and *B*, (Cormen *et al*. 2001), can be expressed as

$$\begin{aligned} R_i &= R_{i-2} - Q_i R_{i-1} \\ R_{i-2} &\geq R_{i-1} \geq R_i \\ i &= 3, 4, \ldots n \end{aligned} \tag{29}$$

i.e. the present reminder, R_i, can be written in terms of the previous two remainders, R_{i-2}, R_{i-1}, and their whole quotient, Q_i. It is also assumed that the reminder in each step of the algorithm can be written as

$$R_i = AX_i + BY_i \tag{30}$$

and the first two reminders are

$$\begin{aligned} R_1 &= A = A1 + B0 \\ R_2 &= B = A0 + B1 \end{aligned} \tag{31}$$

The expression for the last non-zero reminder, $R_n,\ n < \infty$, gives the desired GCD of *A* and *B*, $R_n = D$. The table (matrix) form of the algorithm is

$$\left(\begin{array}{cc|c} 1 & 0 & A \\ 0 & 1 & B \end{array}\right) \sim \begin{array}{c}\text{elementary} \\ \text{row operations}\end{array} \sim \left(\begin{array}{cc|c} Z & W & 0 \\ X & Y & D \end{array}\right) \tag{32}$$

The result satisfies Diophantine equations

$$\begin{aligned} AX + BY &= D \\ AZ + BW &= 0 \end{aligned} \tag{33}$$

Before implementation the extended Euclidean algorithm to the $\mathbf{R}_{MS}$ ring, an *ordering* of elements of the ring has to be defined. For $A(s), B(s) \in \mathbf{R}_{MS}$ holds

1. $A(s) \leq B(s)$ iff $A(s)$ divides $B(s)$.
2. $A(s) = B(s)$ iff $A(s)$ divides $B(s)$ and $B(s)$ divides $A(s)$, or equivalently, $A(s)$ is associated with $B(s)$.
3. $A(s)$ is not related to $B(s)$ iff neither $A(s)$ divides $B(s)$, nor $B(s)$ divides $A(s)$.

Note that $\mathbf{R}_{MS}$ is a *partially ordered set*. The general algorithm always initiates with $A(s) \geq B(s)$, see (29) and (31). Assume these three situations for $\mathbf{R}_{MS}$:

1. If $A(s)$ is associated with $B(s)$, the GCD of both is simply $A(s)$ or $B(s)$.
2. If $A(s) \geq B(s)$, keep the following scheme

$$\left(\begin{array}{cc|c} 1 & 0 & A(s) \\ 0 & 1 & B(s) \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\dfrac{A(s)}{B(s)} & 0 \\ 0 & 1 & B(s) \end{array}\right) \tag{34}$$

hence, GCD of $A(s)$ and $B(s)$ is $B(s)$.

3. Let $A(s)$ and $B(s)$ be not related to each other. In this case, follow this scheme

$$\left(\begin{array}{cc|c} 1 & 0 & A(s) \\ 0 & 1 & B(s) \end{array}\right) \sim$$
$$\sim \left(\begin{array}{cc|c} 1 & -Q(s) & A(s) - Q(s)B(s) \\ 0 & 1 & B(s) \end{array}\right) \sim$$
$$\sim \left(\begin{array}{cc|c} 0 & 1 & B(s) \\ 1 & -Q(s) & A(s) - Q(s)B(s) \end{array}\right) \sim$$
$$\sim \left(\begin{array}{cc|c} \dfrac{-B(s)}{A(s) - Q(s)B(s)} & \dfrac{A(s)}{A(s) - Q(s)B(s)} & 0 \\ 1 & -Q(s) & A(s) - Q(s)B(s) \end{array}\right) \tag{35}$$

In scheme (35), it is supposed that there can be found a quotient $Q(s)$ such that the element $A(s) - Q(s)B(s)$ divides $B(s)$. In other words, the objective is to find a structure of $Q(s)$ and to set zeros and poles of $A(s) - Q(s)B(s)$ such that divisibility conditions as in Subsection 3.3 are satisfied. However, this task can be troublesome, particularly in case of unstable neutral quasipolynomials, due to infinity number of unstable roots. Hence, the question is whether it is *always* possible to prescribe all desired unstable roots – and no other ones – or to prescribe roots of any quasipolynomial so that it is stable. The following example demonstrates this problem and explains scheme (35).

Let these two elements of $\mathbf{R}_{MS}$ be given

$$A(s) = \frac{1 - \exp(-2s)}{s + m_0},\ B(s) = \frac{1 - \exp(-3s)}{s + m_0} \tag{36}$$

Neither $A(s)$ divides $B(s)$, nor $B(s)$ divides $A(s)$. Non-zero zeros of $A(s)$ are

$$s_{A,k,k+1} = \pm k\pi \mathrm{j},\ k \in \mathbf{N} \tag{37}$$

and those of $B(s)$ are

$$s_{B,k,k+1} = \pm \frac{2k\pi}{3}\mathrm{j},\ k \in \mathbf{N} \tag{38}$$

Thus, there exist an infinite number of different unstable zeros of $A(s)$ and $B(s)$. However, there are also infinitely many common unstable zeros

$$s_{k,k+1} = \mathrm{LCM}\{s_{A,k,k+1}, s_{B,k,k+1}\} = \pm 2k\pi,\ k \in \mathbf{N} \tag{39}$$

where LCM{.,.} denotes the least common multiple.

Follow the scheme (35)

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{1-\exp(-2s)}{s+m_0} \\ 0 & 1 & \frac{1-\exp(-3s)}{s+m_0} \end{array}\right) \sim \\ \sim \left(\begin{array}{cc|c} 1 & -Q(s) & \frac{1-\exp(-2s)}{s+m_0} - Q(s)\frac{[1-\exp(-3s)]}{s+m_0} \\ 0 & 1 & \frac{1-\exp(-3s)}{s+m_0} \end{array}\right) \tag{40}$$

Choose $Q(s) = q$ = const. As it can be seen, any option of q does not satisfy the stability of quasipolynomial $1-\exp(-2s)-q[1-\exp(-3s)]$. Hence, first, try to find $Q(s) \in \mathbf{R}_{MS}$ such that

$$\frac{1-\exp(-2s)-Q(s)[1-\exp(-3s)]}{s+m_0} = \frac{1-\exp(-3s)}{s+m_0} \tag{41}$$

This requirement yields

$$Q(s) = \frac{\exp(-3s)-\exp(-2s)}{1-\exp(-3s)} \tag{42}$$

which is not an element of $\mathbf{R}_{MS}$ since some unstable roots of $1-\exp(-3s)$ are different from those of $\exp(-3s)-\exp(-2s)$.

Secondly, try to solve a rather different condition

$$\frac{1-\exp(-2s)-Q(s)[1-\exp(-3s)]}{s+m_0} = \frac{1-\exp(-s)}{s+m_0} \tag{43}$$

As reveals from (38) and (39), $1-\exp(-s)$ divides $1-\exp(-3s)$ since all unstable roots of the former, given by (39), are in the set of unstable roots of the latter. Equation (43) gives

$$Q(s) = \frac{\exp(-s)-\exp(-2s)}{1-\exp(-3s)} \tag{44}$$

One can prove that this expression is already analytic on the imaginary axis.

To finish scheme (35), we have

$$\left(\begin{array}{cc|c} 1 & \frac{\exp(-2s)-\exp(-s)}{1-\exp(-3s)} & \frac{1-\exp(-s)}{s+m_0} \\ 0 & 1 & \frac{1-\exp(-3s)}{s+m_0} \end{array}\right) \sim \\ \sim \left(\begin{array}{cc|c} 0 & 1 & \frac{1-\exp(-3s)}{s+m_0} \\ 1 & \frac{\exp(-2s)-\exp(-s)}{1-\exp(-3s)} & \frac{1-\exp(-s)}{s+m_0} \end{array}\right) \sim \\ \sim \left(\begin{array}{cc|c} -\frac{1-\exp(-3s)}{1-\exp(-s)} & \frac{1-\exp(-2s)}{1-\exp(-s)} & 0 \\ 1 & \frac{\exp(-2s)-\exp(-s)}{1-\exp(-3s)} & \frac{1-\exp(-s)}{s+m_0} \end{array}\right) \tag{45}$$

Hence, the GCD of $A(s)$ and $B(s)$ from (36) is

$$\mathrm{GCD}\{A(s), B(s)\} = \frac{1-\exp(-s)}{s+m_0} \tag{46}$$

Indeed, every common divisor of $A(s)$ and $B(s)$ divides expression (46).

However, there still remains the question of general possibility of construction of a desired quasipolynomial, as stated above.

Note that if the $\mathbf{R}_{MS}$ ring is a Bézout domain (and not a PID), there must exist an infinitely generated ideal which is not principal, thus, the ring is not a *Noetherian ring*.

4.4 *Coprime factorization*

Discuss now the problem of the existence of the coprime factorization for $\mathbf{R}_{MS}$ from a different point of view. The task is, to the given ratio of (quasi)polynomials,

$$G(s) = \frac{b(s)}{a(s)} \tag{47}$$

to find a (quasi)polynomial $m(s)$ such that two elements of the ring, $A(s)$ and $B(s)$, are coprime (relatively prime)

$$G(s) = \frac{b(s)}{a(s)} = \frac{\frac{b(s)}{m(s)}}{\frac{a(s)}{m(s)}} = \frac{B(s)}{A(s)} \tag{48}$$

e.i. there does not exist a non-zero non-unit element $D(s)$ of the ring satisfying

$$A(s) = \overline{A}(s)D(s),\ B(s) = \overline{B}(s)D(s) \tag{49}$$

If both $a(s)$ and $b(s)$ are polynomials, the problem is solved as for the $\mathbf{R}_{PS}$ ring.

Let at least one of $a(s)$ and $b(s)$ be a quasipolynomial instead of a polynomial and the degree of $a(s)$ be obviously greater then or equal to that of $b(s)$. Obviously, $m(s)$ have to be taken of the same degree as $a(s)$. Consider these two cases:

1. Quasipolynomials $a(s)$ and $b(s)$ have no common unstable roots. Then $m(s)$ can be taken as any stable (quasi)polynomial of degree of $a(s)$, as in $\mathbf{R}_{PS}$.
2. Quasipolynomials have some common unstable roots (either finitely or infinitely many). In this case, $m(s)$ have to be found such that it contains all common unstable roots and no other unstable ones. An example analogous to that presented in the previous subsection follows.

Let

$$G(s)=\frac{b(s)}{a(s)}=\frac{1-\exp(-2s)}{1-\exp(-3s)} \tag{50}$$

According to (37) - (39), these quasipolynomials, $a(s)$ and $b(s)$, have an infinite number of common unstable roots. There can be found a quasipolynomial which contains purely these common unstable roots

$$m(s)=1-\exp(-s) \tag{51}$$

so that the coprime factorization is

$$G(s)=\frac{b(s)}{a(s)}=\frac{B(s)}{A(s)}=\frac{\frac{1-\exp(-2s)}{1-\exp(-s)}}{\frac{1-\exp(-3s)}{1-\exp(-s)}} \tag{52}$$

One can easily verify that any other option of $m(s)$ results in a prime ratio, i.e. both quasipolynomials would be divisible by a term with a zero equals to a common unstable root.

Thus the question is the same as for the extended Euclidean algorithm, i.e. whether it is *always* possible to set all desired unstable roots of a quasipolynomial (and not any other unstable ones), even an infinite number of them. On the other hand, the case of common unstable zeros and poles in the plant model function is almost hypothetical.

5 CONCLUSIONS

The present contribution discusses the possibility of introduction of a revisited conception of the $\mathbf{R}_{MS}$ ring utilized for the description of delayed systems. The first part of the paper interferes with the original definition of the ring and suggests some thinks to be thought. There is also proposed a new divisibility condition for the ring here. In the second part of this contribution, some algebraic properties of the new conception are described. It is shown that the ring is neither a unique factorization domain nor a principal ideal domain. The extended Euclidean algorithm for the ring is performed to prove the ring is a Bézout domain. However, this task together with the question of the apriori existence of a coprime factorization remains partially unsolved. Illustration examples rather then exact mathematical definitions and proofs are presented to support the ideas.

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