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OPTIMAL CONTROL OF CHAIN OF INTEGRATORS WITH CONSTRAINTS

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Abstract: Limitations on control signals and magnitudes of state variables as velocity and acceleration are common requirement in motion control applications. It is well known that time optimal control problem for a chain of integrators with bounded input leads to "bang-bang" control strategy and can be converted to finding real solutions of a system of polynomial equations. The similar result is missing for the problem when magnitude constraints on the state of all integrators in the chain except the last one are added. The paper introduces a new solution of this problem which is based on the theory of Gröbner bases.

Keywords: optimal control, Gröbner bases, chain of integrators, "bang-bang" control, constraints

1 INTRODUCTION

The optimal control is widely discussed area in automation. The main motivation for our research was generating and linking of trajectories in the path planning problem. In real applications, especially in motion control of mechatronic system, the control of position, velocity and acceleration is the essential. Quantities mentioned above have mutually differential binding and therefore we can describe them by a chain of integrators with bounded input as a necessary condition in real problems. It is well known that time optimal control problem for the chain of integrators with bounded input leads to "bang-bang" control strategy, see Athans et al. (1966), and can be converted to finding real solutions of a system of polynomial equations, see Walter et al. (2001). The similar result is missing for the modified problem when magnitude constraints on the state of all integrators in the chain except the last one are added. Recently, a new methodology for symbolic manipulation with polynomials has been developed. In this paper the new theory named Gröbner bases is used for solving the time optimal control problem with constraints. In contrast with Discretization methods of dynamic optimization this approach is independent on sampling period and less computational memory intensive. For more on Gröbner approach we refer the reader to Buchberger (1986).

2 STATEMENT OF THE PROBLEM

Consider third order system in the form of the chain integrators with bounded input

$$\begin{bmatrix}\dot{s}\\ \dot{v}\\ \dot{a}\end{bmatrix}=\underbrace{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}s\\ v\\ a\end{bmatrix}}_{x}+\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{B}u \qquad (1)$$

where state variables are position, velocity and acceleration of the system.

Without lost of generality the bound on the control function will be taken to be

$$|u|\le 1 \qquad (2)$$

Furthermore we consider state constraints in the form of magnitudes of velocity and acceleration

$$\begin{aligned} -v_M \leq v(t) \leq v_M \\ -a_M \leq a(t) \leq a_M \end{aligned} \quad (3)$$

If the initial state $x_0 = [a_0, v_0, s_0]$ and the final state $x_f = [a_f, v_f, s_f]$ are given, then these states are related by the expression

$$x_f = e^{A t_f} x_0 + \int_{t_0}^{t_f} e^{A(t_f - \tau)} B u(\tau) d\tau . \quad (4)$$

It is desired to find the time optimal control $u(t)$ which satisfies (2), drives the system from its x_0 to x_f and minimizes the transfer time t_f.

3 BANG BANG CONTROL

It is well known, that using the Maximum Principle of Pontryagin one finds that for the system (1),(2), but without (3), the time optimal control leads to "bang-bang" control with at most *three* time intervals, see Athans et al. (1966).

4 PROBLEM WITH CONSTRAINTS

To the best knowledge of the authors, no similar results exist for time optimal control of constrained system. Although complex constraints can be added to the standard formulation of the Maximum Principle, see Locatelli (2001), the exact solution of the state constrained problem is very complicated. This is why we will accept in the sequel the following hypothesis which provides the possibility to convert the problem of time optimal control of (1),(2),(3) to a system of polynomial equations. The authors believe that this hypothesis can be proved by using Maximum Principle with a global instantaneous inequality constraints.

Hypothesis: *The time optimal control for system (1),(2) with state constraints (3) leads to "bang-null-bang" control with at most seven time intervals t_1-t_7 where input is constant. There are two possible strategies where the input alternates between +1, 0 and -1. Furthermore, single time intervals can vanish if system does not reach the corresponding constraints. Thus the general strategy of the time optimal control can be expressed in two different forms.*

To obtain the simple notation, let us introduce

$$\tau_k = \sum_{i=1}^{k} t_i , \quad (5)$$

where t_i, $i=1,..,7$ are the lengths of the intervals with constant input.

Using (5) the general strategy is defined

$$u_{+(t)} = \begin{cases} 1, t \in [0, \tau_1) \\ 0, t \in [\tau_1, \tau_2) \\ -1, t \in [\tau_2, \tau_3) \\ 0, t \in [\tau_3, \tau_4) \\ -1, t \in [\tau_4, \tau_5) \\ 0, t \in [\tau_5, \tau_6) \\ 1, t \in [\tau_6, \tau_7) \end{cases} \quad (6)$$

$$u_{-(t)} = \begin{cases} -1, t \in [0, \tau_1) \\ 0, t \in [\tau_1, \tau_2) \\ 1, t \in [\tau_2, \tau_3) \\ 0, t \in [\tau_3, \tau_4) \\ 1, t \in [\tau_4, \tau_5) \\ 0, t \in [\tau_5, \tau_6) \\ -1, t \in [\tau_6, \tau_7) \end{cases} \quad (7)$$

If the constraints are both equal to infinity or the system's states do not reach them during the state transfer, then the time optimal control takes the known "bang-bang" form for third order system without state constraints.

Hereafter we restrict the x_0 and x_f of a state movement to the admissible set $\mathbb{R} \times [-v_M, v_M] \times [-a_M, a_M]$. Moreover, it will be assumed that x_0 and x_f belong to the admissible domain in the plane *v-a*, depicted in Fig. 1., where corner curves are given by

$$v = \mp \frac{a^2}{2} \pm v_M , \quad (8)$$

These curves correspond to the trajectories of (1) for the control $u=1,-1$.

Under above assumptions the transfer problem is always feasible and corresponding trajectories, initial and final states are called admissible.

Note, that corner curves (8) define two types of disabled area, see Fig. 1.

1. If initial state x_0 is in initial disabled area, then no control exist for transfer the system without crossing the constraints.
2. If final state x_f is in final disabled area, then no control exist for transfer the system to that point without crossing the constraints.

From Hypothesis it follows that system moves in the plane *v-a* as is depicted in Fig. 2. Fig. 3 shows the evolution of system in time horizon, with all active constraints.

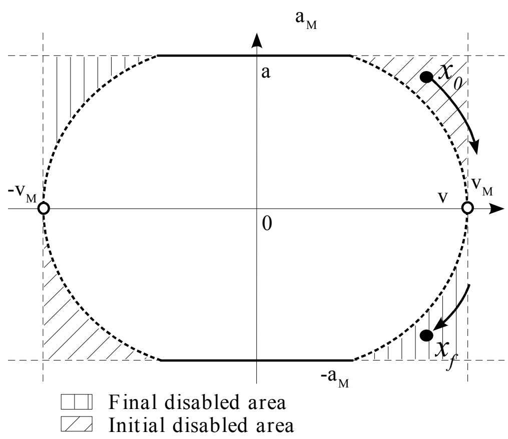

Fig. 1: Projection of admissible domain to the plane *v - a*.

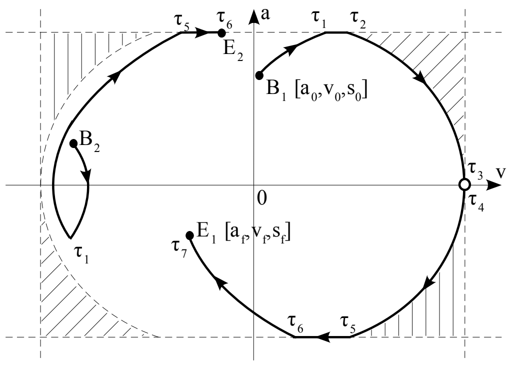

Fig. 2: Two examples of admissible movement of system in the plane v-a

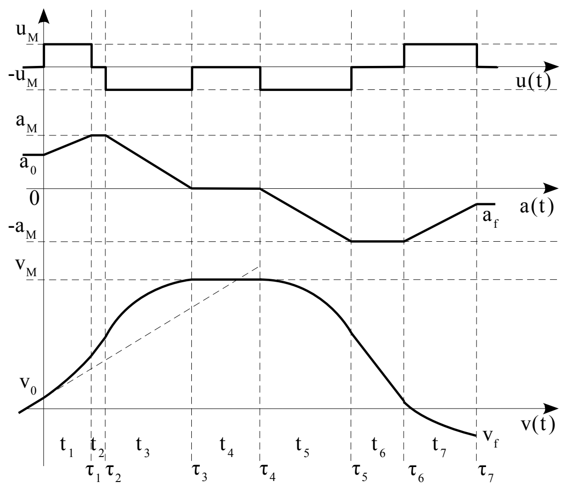

Fig. 3: Time behaviour of one of examples (B_1-> E_1).

5 ALGEBRAIC APPROACH

For finding the switching strategy we use algebraic approach. From Hypothesis there are only two strategies where input alternates as is shown in (6), (7). Taking into account the maximal number of switching, we can find some equations which define the trajectories from the initial state to final state as some functions of time intervals t_i, $i = 1,..,7$.

We compose two sets of equations. One set for u_+, and one for u_-. We restrict ourselves to the case when u_+ is optimal. The process is analogous for u_-. From (4) and (6) we obtain the first three equations.

$$a_f = a_0 + t_1 - t_5 + t_7 - t_3$$

$$\begin{aligned} v_f =& + v_0 + a_0(t_1+t_2+t_3+t_4+t_5+t_6+t_7) \\ & + t_1 t_2 + t_1 t_3 + t_1 t_4 + t_1 t_5 + t_1 t_6 + t_1 t_7 \\ & - t_3 t_4 - t_3 t_5 - t_3 t_6 - t_3 t_7 - t_5 t_6 - t_5 t_7 \\ & - \frac{1}{2} t_3^2 - \frac{1}{2} t_5^2 + \frac{1}{2} t_7^2 + \frac{1}{2} t_1^2 \end{aligned} \tag{8}$$

$$\begin{aligned} s_f =& + s_0 + (t_6+t_1+t_3+t_4+t_2+t_5+t_7) v_0 \\ & + (t_1 t_2 + t_1 t_4 + t_2 t_3 + t_1 t_3 + t_4 t_5 + t_1 t_5 + t_4 t_6) a_0 \\ & + (t_3 t_6 + t_1 t_6 + t_2 t_5 + t_2 t_4 + t_3 t_5 + t_3 t_4 + t_2 t_7) a_0 \\ & + (t_1 t_7 + t_4 t_7 + t_2 t_6 + t_5 t_6 + t_3 t_7 + t_5 t_7 + t_6 t_7) a_0 \\ & + (\frac{1}{2} t_3^2 + \frac{1}{2} t_4^2 + \frac{1}{2} t_1^2 + \frac{1}{2} t_6^2 + \frac{1}{2} t_5^2 + \frac{1}{2} t_7^2 + \frac{1}{2} t_2^2) a_0 \\ & - t_5 t_6 t_7 - t_3 t_4 t_7 - t_3 t_5 t_7 - t_3 t_6 t_7 - t_3 t_4 t_5 - t_3 t_4 t_6 \\ & - t_3 t_5 t_6 + t_1 t_2 t_3 + t_1 t_4 t_7 + t_1 t_3 t_7 + t_1 t_5 t_7 + t_1 t_6 t_7 \\ & + t_1 t_4 t_5 + t_1 t_4 t_6 + t_1 t_3 t_4 + t_1 t_3 t_5 + t_1 t_3 t_6 + t_1 t_2 t_4 \\ & + t_1 t_2 t_5 + t_1 t_2 t_6 + t_1 t_5 t_6 + t_1 t_2 t_7 - \frac{1}{2} t_3^2 t_6 + \frac{1}{2} t_1^2 t_2 \\ & + \frac{1}{2} t_1 t_2^2 + \frac{1}{2} t_1 t_3^2 + \frac{1}{2} t_1^2 t_3 - \frac{1}{2} t_3^2 t_4 - \frac{1}{2} t_3 t_6^2 + \frac{1}{2} t_1 t_4^2 \\ & + \frac{1}{2} t_1^2 t_5 + \frac{1}{2} t_1^2 t_4 + \frac{1}{2} t_1 t_7^2 + \frac{1}{2} t_1^2 t_6 + \frac{1}{2} t_1 t_5^2 + \frac{1}{2} t_1 t_6^2 \\ & + \frac{1}{2} t_1^2 t_7 - \frac{1}{2} t_5 t_6^2 - \frac{1}{2} t_5^2 t_7 - \frac{1}{2} t_5 t_7^2 - \frac{1}{2} t_3 t_5^2 - \frac{1}{2} t_3^2 t_7 \\ & - \frac{1}{2} t_3 t_7^2 - \frac{1}{2} t_3 t_4^2 - \frac{1}{2} t_3^2 t_5 - \frac{1}{2} t_5^2 t_6 \\ & + \frac{1}{6}(-t_3^3 + t_1^3 + t_7^3 - t_5^3) \end{aligned}$$

The possible active constraints define the another four equations.

If the constraint $a(t) \le a_M$ is active then

$$t_1 = a_M - a_0 \tag{9}$$

else

$$t_2 = 0 \tag{10}$$

If the constraint $a(t) \ge -a_M$ is active then

$$t_7 = a_M + a_f \tag{11}$$

else

$$t_6 = 0 \tag{12}$$

Finally, if the constraint $v(t) \leq v_M$ is active then it follows $v(\tau_3) = v(\tau_4) = v_M$ and so

$$\begin{aligned} v_M &= +v_0 + a_0(t_1+t_2+t_3) + t_1 t_2 + t_1 t_3 \\ &\quad -\frac{1}{2}t_3^2 + \frac{1}{2}t_1^2 \\ v_M &= +v_0 + a_0(t_1+t_2+t_3+t_4) + t_1 t_2 + t_1 t_3 \\ &\quad + t_1 t_4 - t_3 t_4 - \frac{1}{2}t_3^2 + \frac{1}{2}t_1^2 \end{aligned} \tag{13}$$

else

$$t_4 = 0 \tag{14}$$

and times t_3 and t_5 can be eliminated and substituted by $t_{35} = t_3 + t_5$

As mentioned above, we have three types of constraint and each constraint can be active or inactive. So we can compose 2^3 systems of seven equations in all. From assumptions mentioned above, we know that one of systems contains real positive solution. Note that finally we will get eight sets of seven equations for u_+ and eight sets for u_-.

6 ALGORITHM

Now we create the computational algorithm for solving systems of equations.

Algorithm

input: x_0, x_f, a_M, v_M

output: $t_1, \ldots, t_7$

step 1. test admissibility of $x_0 = [a_0, v_0, s_0]$ and $x_f = [a_f, v_f, s_f]$. If x_0, x_f are in admissible domain then the solution must exist.
We can check it using vertices of transfer curve

$$v_{max} = \pm v_0 \mp \frac{a_0^2}{2}, \quad v_{max} = \pm v_f \mp \frac{a_f^2}{2}$$

where v_{max} are vertices, and they have to satisfy

$$v_M \geq |v_{max}|$$

step 2. compose the eight systems S_i, $i = 1,2,..,8$ of equations for different possibilities of active constraints. Once for u_+ once again for u_- and insert the x_0 and x_f. See section 5 for details.

step 3. find all solutions using Gröbner bases for each system S_i and take out only real positive solution.

step 4. sort time sequence for each selected solution

$$(t_2, t_3, t_4, t_1, t_6, t_7, t_5) \rightarrow (t_1, t_2, t_3, t_4, t_5, t_6, t_7)$$

step 5. because all combinations of equations do not accept the constraints, check the limits for each time sequence

$$|a(\tau_1)| \leq a_M, |a(\tau_5)| \leq a_M, |v(\tau_3)| \leq v_M$$

step 6. if still exist more than one real positive solution, take out that one which minimizes

$$\sum_{i=1}^{7} t_i = \tau_7 = t_f$$

7 RESULTS OF ALGORITHM

In this section we examine some results of algorithm. First we set x_0, x_f and constraints (3) so that all time intervals will be nonzero.

$$\begin{aligned} &a_0 = 1, v_0 = 1/2, s_0 = -2 \\ &a_f = -1/2, v_f = 1, s_f = 20 \\ &a_M = 3/2, v_M = 4 \end{aligned}$$

Resultant time intervals for time optimal control from x_0 to x_f with state constraints and bounded input are

$$\begin{aligned} &t_1 = 0.5\,s, \quad t_2 = 1.166666667\,s \\ &t_3 = 1.5\,s, \quad t_4 = 1.389322917\,s \\ &t_5 = 1.5\,s, \quad t_6 = 0.5833333333\,s \\ &t_7 = 1\,s \end{aligned}$$

Trajectory of system in state space is depicted in Fig.4., and according to *Hypothesis* (Fig.2.) we can easily recognize the projection of trajectory to the plane *v-a* depicted in Fig. 5. Figure 6. shows the evolution of each state variable in time horizon.

Second example shows the situation in which state of the system does not reach the constraints. Therefore the appropriate resultant time intervals are equal to zero. x_0, x_f and constraints are given in (15). Fig. 7. and Fig. 8. display the simulation results for this example.

$$\begin{aligned} &a_0 = 1, v_0 = 1/2, s_0 = -2 \\ &a_f = -1/2, v_f = 1, s_f = 3 \\ &a_M = 3/2, v_M = 4 \end{aligned} \tag{15}$$

Resultant time intervals for this example are

$$\begin{aligned} &t_1 = 0.39427\,s, \quad t_2 = 0\,s \\ &t_3 = 1.39427\,s, \quad t_4 = 0\,s \\ &t_5 = 1.03391\,s, \quad t_6 = 0\,s \\ &t_7 = 0.53391\,s \end{aligned} \tag{16}$$

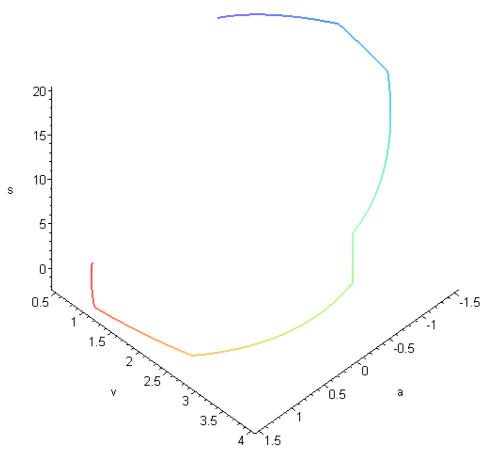

Fig. 4: Transfer trajectory of system in state space. All constraints are active.

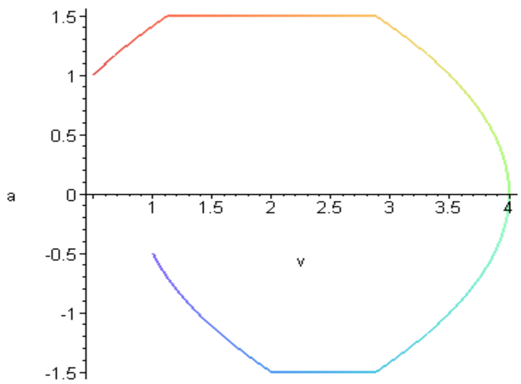

Fig. 5: Transfer trajectory of system in plane *v-a*. All constraints are active.

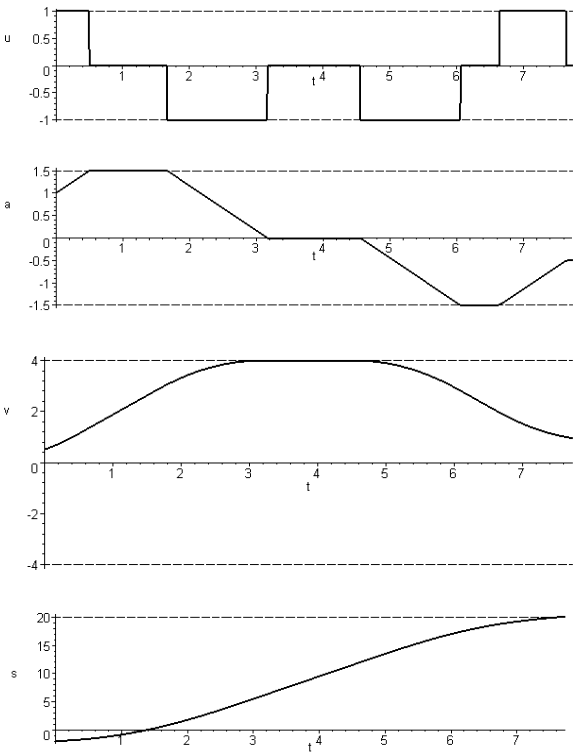

Fig. 6: Evolution of state variables in time horizon with all active constraints.

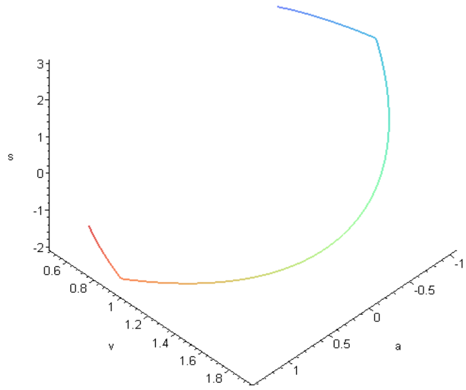

Fig. 7: Transfer trajectory of system in state space. All state constraints are inactive.

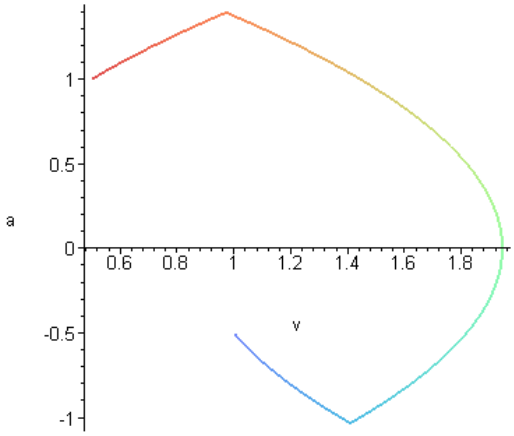

Fig. 8: Transfer trajectory of system in plane *v-a*. All state constraints are inactive.

8 CONCLUSION

In this paper we used the theory of Gröbner bases to study the problem of time optimal control for the chain of integrators with state constraints and bounded input. We gave a brief overview of some facts from time optimal control for this problem. We used the *Hypothesis* following from Maximum Principle to find systems of polynomial equations for time switching intervals. Using Gröbner bases to derive the systems of polynomial equations we designed the algorithm for solving the constrained time optimal problem. We gave a examples of constrained and unconstrained problem that captures the main features of the theoretical result.

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