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DISCRETE-TIME SOLUTION TO THE DISTURBANCE DECOUPLING PROBLEM OF COUPLED TANKS

Miroslav Halás

Institute of Control and Industrial Informatics, Fac. of Electrical Engineering and IT, Slovak University of Technology
Ilkovičova 3, 81219 Bratislava, Slovakia
e-mail : miroslav.halas@stuba.sk

Abstract: Mathematical technicalities, involved in the modern theory of non-linear control systems, many times prevent a wider use of the impressive theoretical results in practice. Attempts to overlap this gap between theory and practice are usually more than welcome and form the main scope of our interest in this work. An important control problem given by the disturbance decoupling is studied for a real laboratory model of coupled tanks. Since the theoretical solution to the disturbance decoupling problem does not satisfy practical control requirements it is modified accordingly. Experiments on the real plant are included as well and show that the disturbances practically do not affect the system output.

Keywords: nonlinear discrete-time systems, applications, algebraic methods, disturbance decoupling, coupled tanks

1. INTRODUCTION

The modern theory of nonlinear control systems all, continuous-time, discrete-time and time-delay, owes a large part of its succes to the systematic use of differential algebraic methods. Since early 80's of the last century this has been forming the scope of interest of many authors in a number of works, see for instance Fliess (1985 1992); Conte et al. (1993); Aranda-Bricaire et al. (1995 1996); Kotta et al. (2001); Xia et al. (2002); Conte et al. (2007) and references therein. Nowadays, such methods offer solutions to a wide range of nonlinear control problems including feedback linearization, model matching, disturbance decoupling, realization problem, non-interacting control, observer design and many others.

However, a price one has to pay for such impressive and elegant solutions is given by a necessity to involve many mathematical technicalities. Obviously, this prevents a wider use of the theoretical results in practice, making the big gap between control theory and control practice even bigger in this case. It is generally known that in practice the way of dealing with nonlinear control systems is many times based just on the linearization in a fixed operating point and then methods of linear control systems are applied. Therefore, attempts to overlap the gap are usually more than welcome and form the main scope of our interest in this work. In particular, an important control problem given by the disturbance decoupling, which is quite frequent control problem in practice, is studied. We begin with the theoretical solutions of Conte et al. (2007) and apply them to the laboratory model of coupled tanks, which is a demonstrative and well know system having contact points to many real control processes, for instance from chemical engineering. It is shown that the theoretical solutions cannot be directly applied and additional problems, related for instance to the difference between model and real system, have to be considered as well. Similar solution as discussed in this paper has recently been given in Žilka and Halás (2010) for continuous-time case, while here the discrete-time counter part is treated. Certain contact points exist also to the non-interacting problem studied in Halás and Žilka (2011). Finally, for additional existing results of the disturbance decoupling problem for nonlinear discrete-time systems the reader is referred for instance to Kotta (1995) where a simple inversion-based solution is given and to Grizzle (1985) where a more advanced differential geometric solution can be found.

2. DISTURBANCE DECOUPLING

We begin with an introduction to the disturbance decoupling problem of nonlinear control systems as discussed in Conte et al. (2007) to which the reader is referred for additional details and references. The ideas can easily be carried over to the discrete-time systems.

For the sake of simplicity we introduce the following notation. For any variable $\xi(t)$ we write only ξ and for its time shifts $\xi(t+T)$, $\xi(t+2T)$ we write ξ^+, ξ^{++} respectively, or, in general, $\xi^{[k]}$ for $\xi(t+kT)$, where T is a sampling period.

Using the above introduced notation the systems considered in this paper are objects of the form

$$\begin{aligned} x^+ &= f(x,u) \\ y &= g(x) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}$ and entries of f and g are meromorphic functions from the difference field denoted by $\mathcal{K}$. For more details see Aranda-Bricaire et al. (1996); Kotta et al. (2001); Halás et al. (2009).

In the disturbance decoupling our task is to design, if possible, a control law such that the disturbances do not affect the system output. Technically speaking, the solution consists of finding a feedback under which a subspace of the state space, affected by disturbances, becomes unobservable in the compensated system. This situation can be explained by the following introductory system

$$\begin{aligned} x_1^+ &= x_2 u \\ x_2^+ &= w \\ y &= x_1 \end{aligned}$$

where w is the disturbance.

As can be seen, through x_2 the disturbance w affects the system output

$$y^{++} = u^+ w$$

However, the state feedback $u = v/x_2$, with v being an input to the compensated system, makes x_2 unobservable in the compensated system and thus decouples the disturbance w from the system output

$$y^+ = v$$

The general solution follows the same idea. That is, if possible, make unobservable the subspace of the state space affected by the disturbance.

Problem statement. Consider the SISO system

$$\begin{aligned} x^+ &= f(x,u) + p(x)w \\ y &= g(x) \end{aligned}$$

where the state $x \in \mathbf{R}^n$, the disturbance $w \in \mathbf{R}^q$ and the entries of f, g and p are elements of the difference field of meromorphic functions $\mathcal{K}$. Find, if possible, a static state feedback

$$u = \alpha(x, v)$$

such that

$$\mathrm{d}y^{[i]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}v, \ldots, \mathrm{d}v^{[i]}\}$$

for any $i \in \mathbf{N}$.

Theorem 1. Let $\mathcal{X} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$ and $\mathcal{Y} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{[i]}; i \geq 0\}$. The disturbance decoupling problem is solvable if and only if $p(x) \perp \mathcal{X} \cap \mathcal{Y}$.

PROOF. The proof follows the same line as in Conte et al. (2007), however, carried over to the discrete-time case.

3. COUPLED TANKS

Coupled tanks are well-known and illustrative system having contact points to many real control processes. For that reason practically each laboratory which activities are related to the system and control theory possesses such a plant. In this section, the mathematical model of the laboratory plant is built up, from its identification to the nonlinear discrete-time state-space model. Then, the disturbance decoupling is applied.

3.1 *System identification*

We restrict our attention to a standard coupled two-tank system, however, with all three valves active. The structure of such a system is depicted in Fig. 1. Our aim is to control the level in the first tank which is, however, coupled with the second tank by a valve with the flow coefficient c_{12}. Each of the tanks is equipped by a valve itself, having the flow coefficients c_1 and c_2 respectively. However, the valve c_2 is considered here as the disturbance w. Thus, we deal here with a SISO

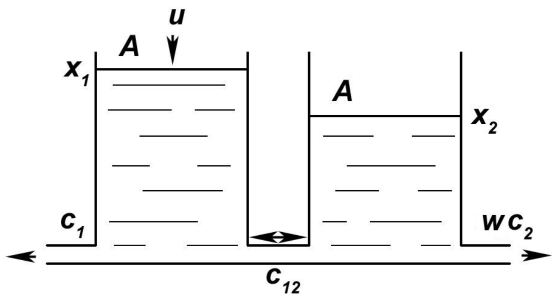

Fig. 1. Coupled tanks

system which can be modelled by the following state-space equations

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{A}u - c_{12}\,\mathrm{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} - c_1\sqrt{x_1} \\
\dot{x}_2 &= c_{12}\,\mathrm{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} - wc_2\sqrt{x_2} \\
y &= x_1
\end{aligned} \tag{2}
$$

where x_1 and x_2 are levels in tank 1 and tank 2 respectively and A is a cross-section of the tanks, see Fig. 1. Note that both tank 1 and tank 2 have the identical cross-sections here. The disturbance $w \in \{0, 1\}$, depending on whether the valve c_2 is switched off or on respectively. In this case the level of a liquid in tank 2 might be greater than in tank 1. For that reason a more general model (2) has to be used.

To identify the system we have to find, besides the cross-sections A, the values of flow coefficients c_1, c_2 and c_{12}. The usual methods to treat the identification are based on applying a couple of certain experiments and measurements. Then the coefficients are computed by using either the steady-states of the system or the system linearization in a fixed operating point. However, both of them are rather slow. In addition, it is, usually, recommended to find a set of values in different steady states or operating points respectively and take their average finally. For that reason we, in what follows, suggest a different approach to the system identification which is based on finding a solution to the reduced nonlinear differential equations of the system (2). As a result we will be able to compute all the coefficients only by measuring the time of the respective experiments.

To identify the flow coefficient c_1 suppose that all valves are closed and the pump is inactive. Let $x_{10} \neq 0$ be a level of a liquid in tank 1. The experiment consists of opening the valve c_1 only and measuring the time τ that it takes to empty the tank from an initial value x_{10} to a final value x_{11}. Obviously, this situation can be modelled by the reduced nonlinear differential equation

$$\dot{x}_1 = -c_1\sqrt{x_1}$$

Even though the equation is nonlinear the solution can easily be found as follows

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -c_1\sqrt{x_1} \\
\int\limits_{x_{10}}^{x_{11}} \frac{1}{\sqrt{x_1}}\mathrm{d}x_1 &= -\int\limits_{0}^{\tau} c_1\mathrm{d}t \\
\Big[2\sqrt{x_1}\Big]_{x_{10}}^{x_{11}} &= \big[-c_1 t\big]_0^\tau \\
2(\sqrt{x_{11}} - \sqrt{x_{10}}) &= -c_1\tau
\end{aligned}
$$

Finally

$$c_1 = \frac{2(\sqrt{x_{10}} - \sqrt{x_{11}})}{\tau}$$

If the final value x_{11} is chosen to be 0, which is usually the most reasonable choice, then the formula reduces to

$$c_1 = \frac{2\sqrt{x_{10}}}{\tau} \tag{3}$$

where τ is the time it takes to empty tank 1 completely from the initial value x_{10}.
Clearly, the analogous experiment can be repeated for the second tank giving us the formula

$$c_2 = \frac{2\sqrt{x_{20}}}{\tau} \tag{4}$$

where this time τ is the time it takes to empty tank 2 completely from its initial value x_{20}.
To identify the flow coefficient c_{12} a more advanced experiment is needed. Suppose that all valves are closed and both pumps inactive. Let $x_{10} \neq 0$ be a level of a liquid in tank 1 and $x_{20} < x_{10}$ be a level of a liquid in tank 2. This time the experiment consists of opening the valve c_{12} only and measuring the time τ it takes the level in tank 1 decrease from the initial value x_{10} to the final value x_{11}. Such a situation can be modelled by the following nonlinear differential equations

$$
\begin{aligned}
\dot{x}_1 &= -c_{12}\sqrt{x_1 - x_2} \\
\dot{x}_2 &= c_{12}\sqrt{x_1 - x_2}
\end{aligned}
$$

However, one can use either of them to find the solution. For instance the first equation yields

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -c_{12}\sqrt{x_1 - x_2} \\
\frac{1}{\sqrt{x_1 - x_2}}\mathrm{d}x_1 &= -c_{12}\mathrm{d}t
\end{aligned}
$$

Note that the situation during the experiment implies that $x_{10} - x_1 = x_2 - x_{20}$ and thus substituting $x_2 = x_{10} + x_{20} - x_1$ gives us

$$
\begin{aligned}
\int\limits_{x_{10}}^{x_{11}} \frac{1}{\sqrt{2x_1 - x_{10} - x_{20}}}\mathrm{d}x_1 &= -\int\limits_{0}^{\tau} c_{12}\mathrm{d}t \\
\Big[\sqrt{2x_1 - x_{10} - x_{20}}\Big]_{x_{10}}^{x_{11}} &= \big[-c_{12} t\big]_0^\tau
\end{aligned}
$$

$$\sqrt{2x_{11} - x_{10} - x_{20}} - \sqrt{x_{10} - x_{20}} = -c_{12}\tau$$

Finally

$$c_{12} = \frac{\sqrt{x_{10} - x_{20}} - \sqrt{2x_{11} - x_{10} - x_{20}}}{\tau}$$

Here, if the initial value x_{20} is chosen to be 0 and the final value x_{11} is chosen to be $x_{10}/2$, that is the levels in both tanks finally equal each other (note that the tanks have the identical cross-sections) the formula reduces to

$$c_{12} = \frac{\sqrt{x_{10}}}{\tau} \tag{5}$$

where τ is the time it takes the levels in both tanks equal each other.

Using the above formulas (3), (4) and (5) the flow coefficients c_1, c_2 and c_{12} of the laboratory plant were identified as $1.17 \cdot 10^{-2}$, $1.17 \cdot 10^{-2}$ and $2.65 \cdot 10^{-2}$ respectively. Finally, the cross sections of both tanks are approximately $A = 10.18 \cdot 10^{-4} m^2$.

3.2 *Discrete-time state-space representation*

To find a discrete-time state-space representation of the system (2) one needs to find a solution to the set of nonlinear differential equations and sample it by the sampling period T. Since the system equations (2) involve nonlinear functions we are, in general, not able to find any. In such a case one usually has to rely on approximations only. One of the possibilities is to employ the Taylor series expansion.
Assume that

$$\dot{x}(t) = f(x(t), u(t))$$

where f is analytic. Then one can write

$$x(t+T) = x(t) + \dot{x}(t)T + \frac{\ddot{x}(t)}{2!}T^2 + \frac{x^{(3)}(t)}{3!}T^3 + \cdots$$

However, it is usually sufficient to consider only the first two terms of the Taylor series expansion to approximate the system behaviour in which case one gets the well-known Euler approximation

$$\dot{x}(t) \approx \frac{x(t+T) - x(t)}{T}$$

Using such an approximation one can find the discrete-time state-space model from (2) as

$$\begin{aligned} x_1^+ &= x_1 + \frac{T}{A}u - c_{12}T \operatorname{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} \\ &\quad -c_1 T\sqrt{x_1} \\ x_2^+ &= x_2 + c_{12}T \operatorname{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} \\ &\quad -wc_2 T\sqrt{x_2} \\ y &= x_1 \end{aligned} \tag{6}$$

3.3 *Disturbance decoupling problem*

The standard theoretical solution to the disturbance decoupling problem, as outlined in Section 2, does not meet basic practical control requirements, as shown in what follows, and thus it is necessary to modify it accordingly.
To proceed with the disturbance decoupling we compute

$$\begin{aligned} y^+ &= x_1 + \frac{T}{A}u - c_{12}T \operatorname{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} \\ &\quad -c_1 T\sqrt{x_1} \end{aligned}$$

Since y^+ directly depends on the input u, that is the relative degree of the system is 1, and it is not affected by the disturbance w, it can be decoupled. Note that in according to Theorem 1 we have $\mathcal{X} \cap \mathcal{Y} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_1\}$ and thus $p(x) = (0, -c_2\sqrt{x_2})^T$ is orthogonal to $\mathcal{X} \cap \mathcal{Y}$.
By solving for u the equation

$$y^+ = v$$

one gets

$$\begin{aligned} u &= \frac{A}{T}v - \frac{A}{T}x_1 + c_{12}A \operatorname{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} \\ &\quad +c_1 A\sqrt{x_1} \end{aligned} \tag{7}$$

where v represents input to the compensated system which is reduced to the first order linear system $y^+ = v$ with the transfer function

$$F(z) = \frac{1}{z} \tag{8}$$

However, from a practical point of view the compensated system cannot respond in one sampling period T, like its transfer function (8) says, at least for lower sampling periods T, for we have a controller output constraint $u \in \langle 0, q_{max} \rangle$ where q_{max} is upper limit of the pump capacity. On the other side for higher sampling periods T the discrete-time approximation (6) of the continuous-time system (2) might no longer be sufficient. In addition, there obviously exist additional differences between the real plant and its continuous-time model (2) that have not been considered. For that reason, the real compensated system will possess oscillations even for not that high sampling periods T when the discrete-time approximation (6) is still accurate. Last but not least, the feedback (7) is not a controller at all. Obviously, it is only a static state feedback achieving the disturbance decoupling, however, with no intention to track the reference signal or to eliminate unmodelled disturbances. For all the aspects listed above, such a solution is practically not applicable and needs to be modified accordingly.

There exist several possibilities how to overcome the problems. One of them, discussed in Žilka and

Halás (2010) for continuous-time case, suggests to modify the feedback (7) such that the whole second tank, together with the disturbance, becomes unobservable. Then, one only has to design a controller for a one-tank system which is, obviously, a trivial task and plenty of solutions have been given. This seems to be a reasonable choice also in the discrete-time case. The feedback (7) can easily be modified to the form

$$u = v + c_{12}A\,\mathrm{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} \quad (9)$$

under which the compensated system takes now the form of one-tank system with the discrete-time state-space model

$$\begin{aligned} x_1^+ &= x_1 + \frac{T}{A}v - c_1T\sqrt{x_1} \\ y &= x_1 \end{aligned} \quad (10)$$

Then the controller can easily be designed by the system linearization in a fixed operating point (x_{10}, v_0, y_0) which reads

$$\begin{aligned} \Delta x_1^+ &= \Delta x_1 + \frac{T}{A}\Delta v - \frac{c_1T}{2\sqrt{x_{10}}}\Delta x_1 \\ \Delta y &= \Delta x_1 \end{aligned}$$

where $\Delta x_1 = x_1 - x_{10}$, $\Delta v = v - v_0$ and $\Delta y = y - y_0$. It has the transfer function

$$F(z) = \frac{K}{z - D}$$

where $K = T/A$ and $D = 1 - \frac{c_1T}{2\sqrt{x_{10}}}$.
If one wants the transfer function of the compensated system

$$G(z) = \frac{R(z)F(z)}{1 + R(z)F(z)}$$

to take the form of a first order linear system with the time constant T_1, then the solution is given by a linear discrete-time PI controller with the transfer function

$$R(z) = \frac{(1-\lambda)(z-D)}{K(z-1)} = \frac{1-\lambda}{K}\left(1 + \frac{1-D}{z-1}\right)$$

where $\lambda = e^{-T/T_1}$.

Remark 2. Note that more advanced solution, dealing also with the system linearization, the controller output constraint and two different disturbances to decouple, has been suggested in Žilka et al. (2009).

The closed loop structure is depicted in Fig. 2. The responses of the real laboratory plant are shown in Fig. 3 where one can observe the differences between the linear PI-controller with and without the disturbance decoupling (9). In the latter the disturbances practically do not affect the system output. However, since we have the constrained controller output and since only a standard PI-controller has been used to control the system both solutions admit an overshoot. A non-overshooting solution has been suggested in Žilka and Halás (2010).
The parameters were chosen as follows: $T = 0.25s$, $T_1 = 5s$ and $x_{10} = 0.2m$.

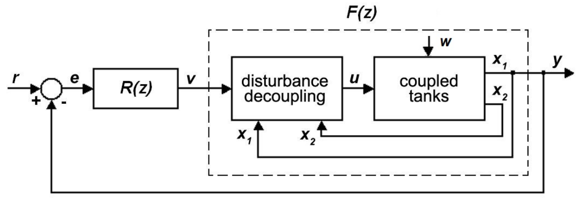

Fig. 2. Closed loop

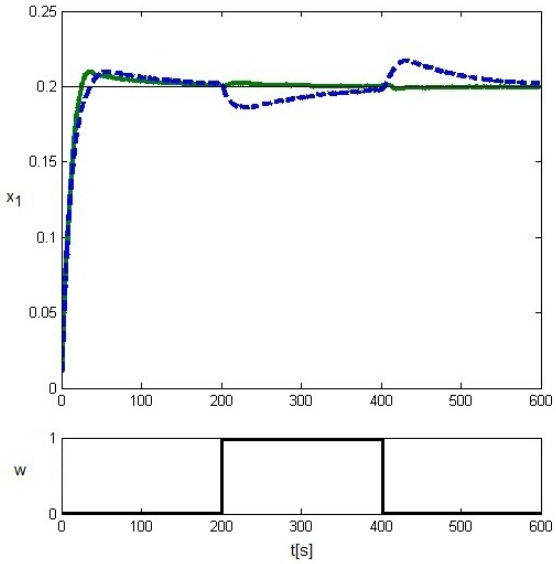

Fig. 3. Closed loop responses: PI-controller with (solid, green line) and without (dashed, blue line) the disturbance decoupling.

Our final note is related to the slight modification of the disturbance decoupling (9) which is appropriate from a practical point of view and has been implemented in our solution. When the valve c_2 is closed the equations (2) imply that in a steady state one, theoretically, has $x_1 = x_2$. However, in practice, there are differences between x_1 and x_2 caused at least by sensors calibration and noise. Therefore the term $\mathrm{sign}(x_1 - x_2)$ in (9) oscillates between 1 and -1 and thus produces small oscillations of the controller output especially in steady states, which is, of course, inconvenient. The problem can and has been overcome easily by adding a deadzone to the disturbance decoupling making it inactive whenever the difference between x_1 and x_2 is less than $2 \cdot 10^{-3}m$.

4. CONCLUSIONS

In this work, an attempt to overlap the gap between control theory and control practice was studied. An important practical control problem given by the disturbance decoupling problem were applied on coupled tanks. It was shown that the initial theoretical solution to the disturbance decoupling problem does not satisfy the basic control requirements. For that reason, the solution was modified accordingly. This resulted in the *PI*-controller with the disturbance decoupling. As a result the disturbances practically did not affect the system output of the real laboratory plant.

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