Theory in Model Predictive Control : Constraint Satisfaction and Stability

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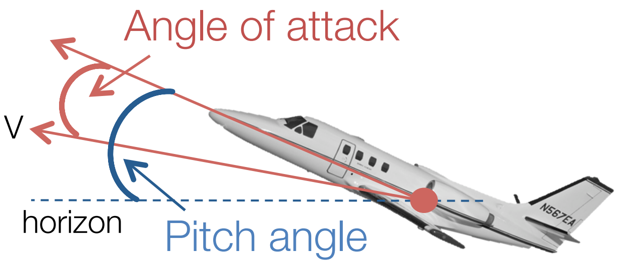
Example: Cessna Citation Aircraft

Linearized continuous-time model:

(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle $\pm 0.262 rad$ ($\pm 15^o$), elevator rate $\pm 0.524 rad$ ($\pm 60^o$)
  pitch angle ± 0.349 ($\pm 30^o$)

Open-loop response is unstable (open-loop poles: $0, 0, -1.5594 \pm 2.2900i$)

[J. Maciejowski, Predictive Control with constraints, 2002]

LQR and Linear MPC with quadratic cost

- Quadratic performance measure
- Linear system dynamics $x^+ = Ax + Bu$
  $y = Cx + Du$
- Linear constraints on inputs and states

LQR:

$$
\begin{aligned}
J^\infty(x) = \min_{x,u} \sum_{i=0}^{\infty} & x_i^T Q x_i + u_i^T R u_i \\
\text{s.t. } x_{i+1} &= Ax_i + Bu_i \\
x_0 &= x
\end{aligned}
$$

MPC:

$$
\begin{aligned}
J^*(x) = \min_{x,u} \sum_{i=0}^{N-1} & x_i^T Q x_i + u_i^T R u_i \\
\text{s.t. } x_{i+1} &= Ax_i + Bu_i \\
x_0 &= x \\
Cx_i + Du_i &\leq b
\end{aligned}
$$

Assume: $Q = Q^T \succeq 0,\ R = R^T \succ 0$

MPC problem can be translated into a quadratic program (QP)

Linear MPC with linear costs

- Linear performance measure (e.g. economically motivated cost, errors)
- Linear system dynamics $x^+ = Ax + Bu$

$$y = Cx + Du$$

- Linear constraints on states and inputs

Resulting MPC problem:

$$\begin{aligned} J^*(x) = \min_{\mathrm{x,u}} \quad & \sum_{i=0}^{N-1} \|Qx_i\|_p + \|Ru_i\|_p \\ \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\ & Cx_i + Du_i \leq b \\ & x_0 = x \end{aligned}$$

Optimization problem can be translated into a linear program (LP) for $p=1/\infty$.

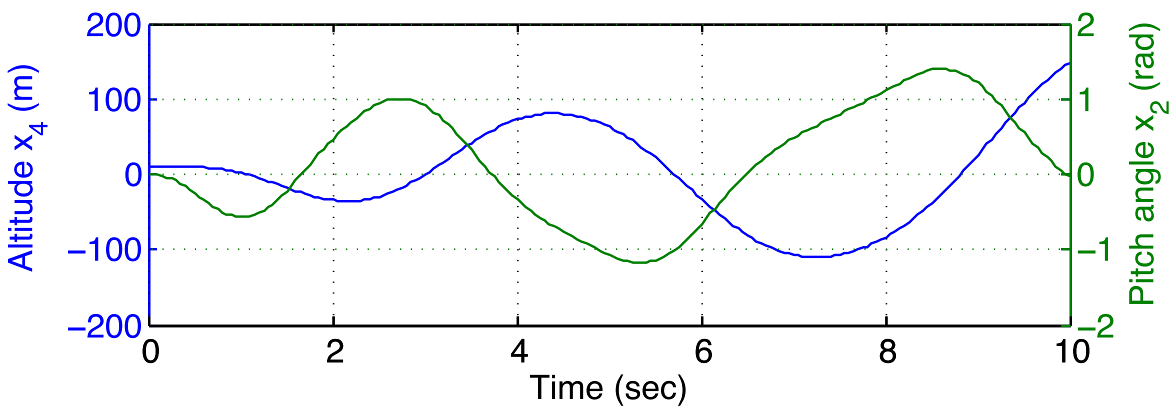
Example: Cessna Citation Aircraft LQR with saturation

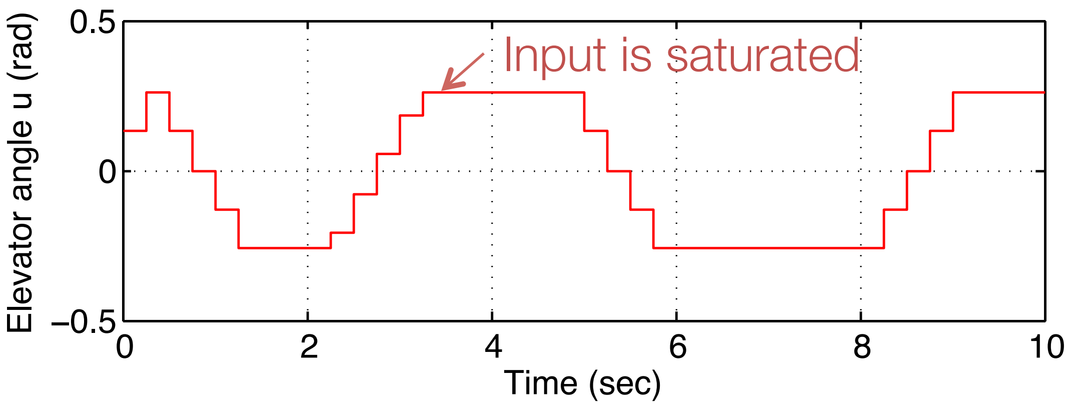
Linear quadratic regulator with saturated inputs:

At time t = 0 the plane is flying with a deviation of 10m of the desired altitude, i.e. x (0)=[0; 0; 0; 10]

Problem parameters:

Sampling time $T_s=0.25sec$, $Q=I$, $R=10$





→ Closed-loop system is unstable

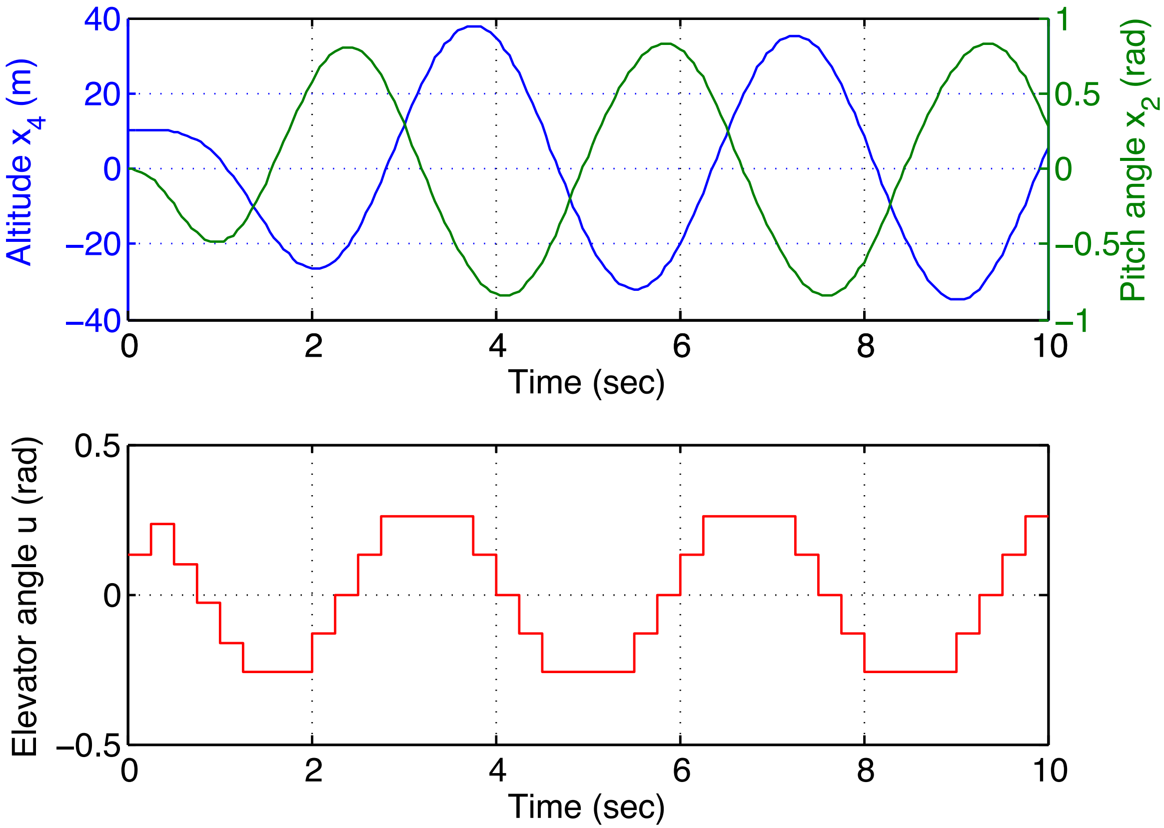
→ Applying LQR control and saturating the controller can lead to instability…

Example: Cessna Citation 500 aircraft
MPC with bound constraints on inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Problem parameters:

Sampling time $T_s = 0.25 sec$,
$Q = I$, $R = 10$, $N = 10$



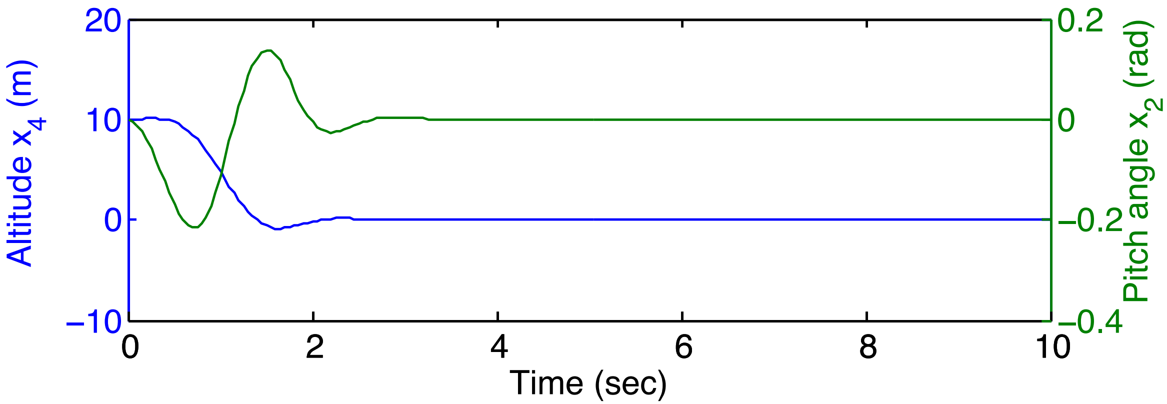
The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

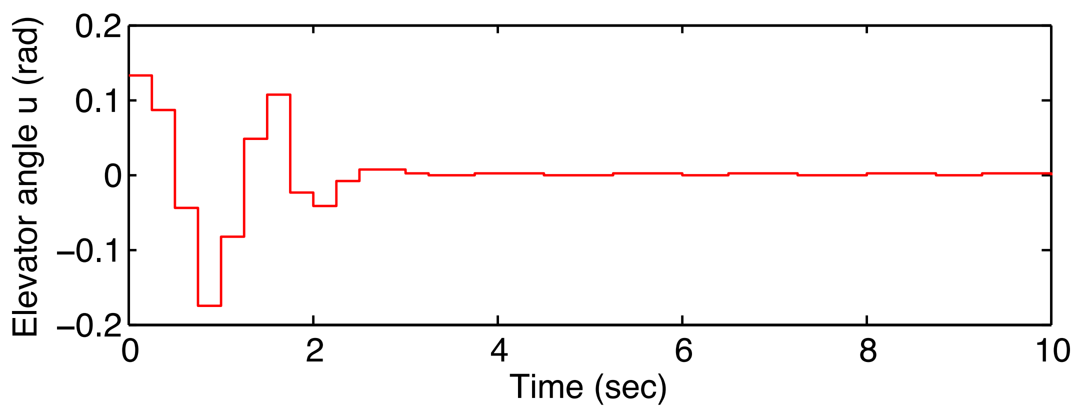
→ System does not converge to desired steady-state but to a limit cycle

Example: Cessna Citation Aircraft
MPC with all input constraints

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$, $u_{-1} = u_{prev}$

Problem parameters:
Sampling time $T_s = 0.25 sec$,
$Q = I$, $R = 10$, $N = 10$





The MPC controller considers all constraints on the actuator

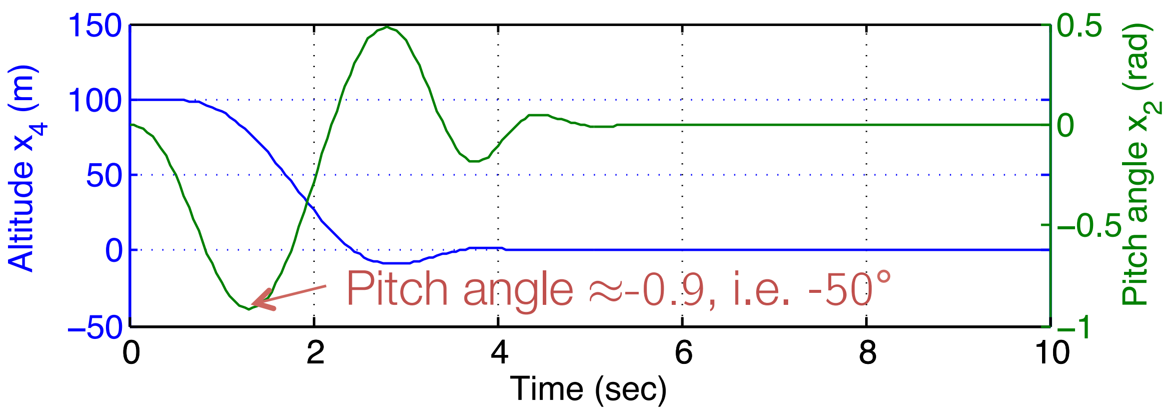
→ Closed-loop system is stable!
→ Efficient use of the control authority

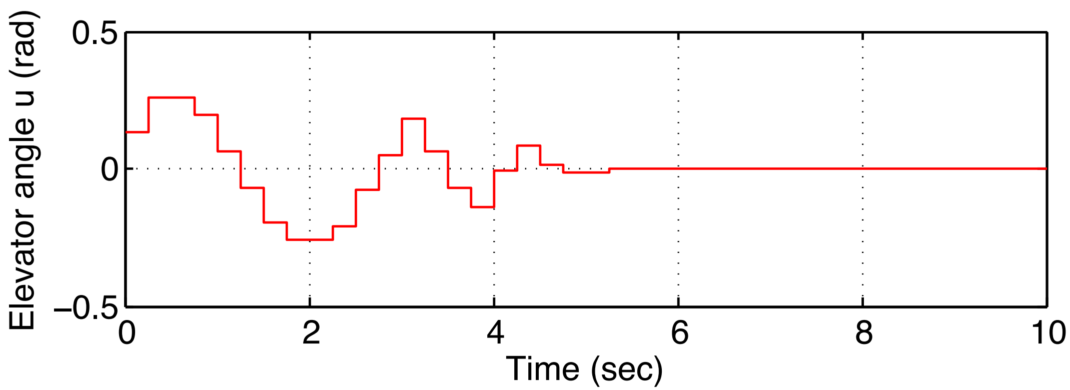
Example: Cessna Citation Aircraft
Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$, $u_{-1} = u_{prev}$

Problem parameters:

Sampling time T_s=0.25sec,
Q=I, R=10, N=10





Increase step:
At time t = 0 the plane is flying with a deviation of 100m of the desired altitude,
i.e. x (0)=[0; 0; 0; 100]

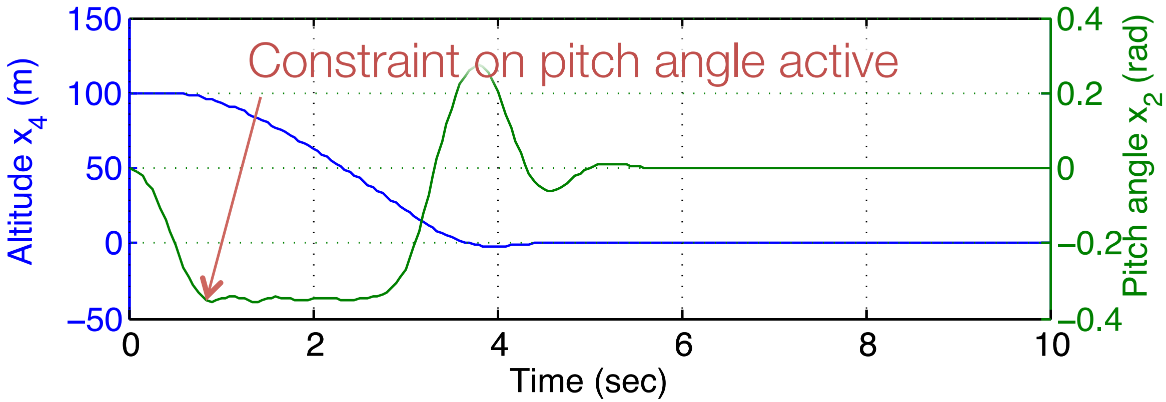
→ Pitch angle too large during transient

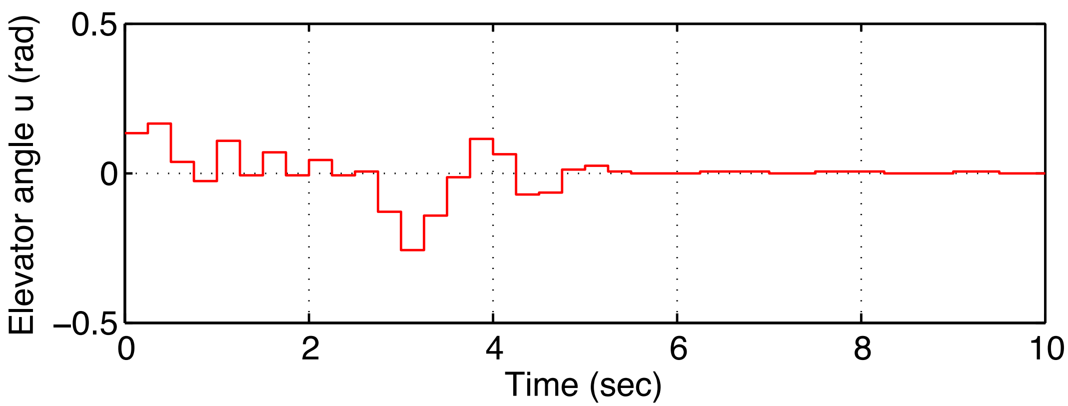
Example: Cessna Citation Aircraft
Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$, $u_{-1} = u_{prev}$

Problem parameters:

Sampling time $T_s = 0.25 sec$,
$Q = I$, $R = 10$, $N = 10$





Add state constraints for passenger comfort:

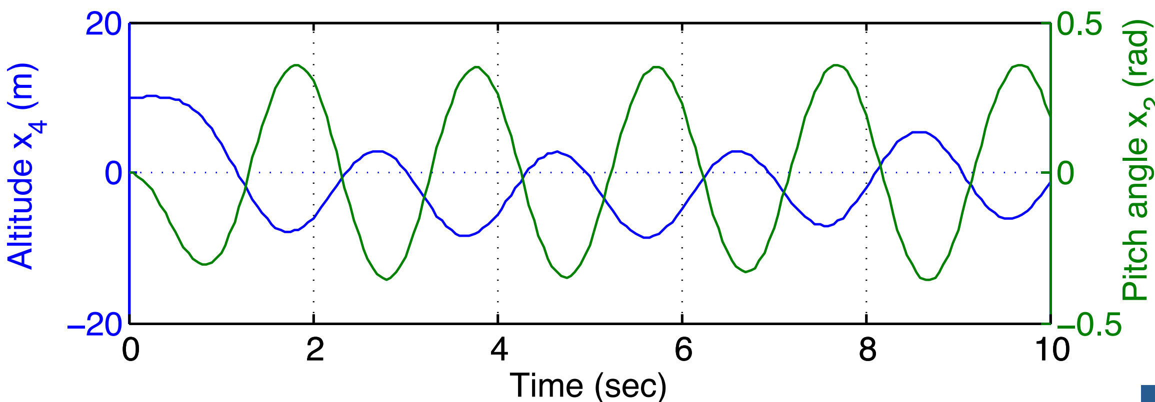
$$|x_2| \leq 0.349$$

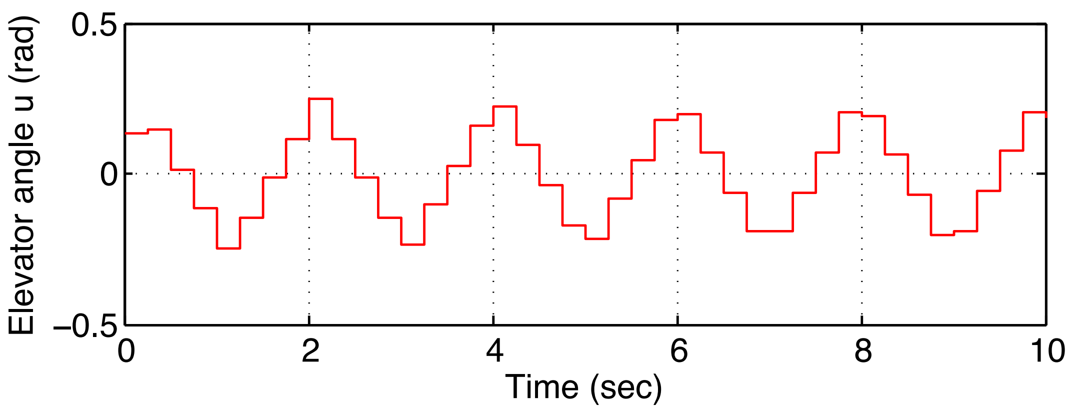
Example: Cessna Citation Aircraft
Shorter horizon causes loss of stability

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$, $u_{-1} = u_{prev}$

Problem parameters:
Sampling time $T_s = 0.25 sec$,
$Q = I$, $R = 10$, $N = 4$

Decrease in the prediction horizon causes loss of the stability properties





This part of the workshop:
How can constraint satisfaction and stability in MPC be guaranteed?

Outline

- Motivating Example: Cessna Citation Aircraft
- Constraint satisfaction and stability in MPC
  - Main idea
  - Problem setup for the linear quadratic case
- How to prove constraint satisfaction and stability in MPC
- Implementation
- Theory extensions

Loss of feasibility and stability guarantees

What can go wrong with standard MPC approach?

→ No feasibility guarantee, i.e. the MPC problem may not have a solution

→ No stability guarantee, i.e. trajectories may not converge to the origin

$$\begin{aligned} \min_{x,u} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\ & x_0 = x \\ & C x_i + D u_i \leq b \end{aligned}$$

Definition: Feasible set

The *feasible set* $\mathcal{X}_N$ is defined as the set of initial states x for which the MPC problem with horizon N is feasible, i.e.

$$\mathcal{X}_N := \{x \mid \exists [u_0, \ldots, u_{N-1}] \text{ such that } C u_i + D x_i \leq b, i = 1, \ldots, N\})$$

Example: Loss of feasibility

Consider the double integrator: $x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

subject to the input constraints $-0.5 \leq u \leq 0.5$

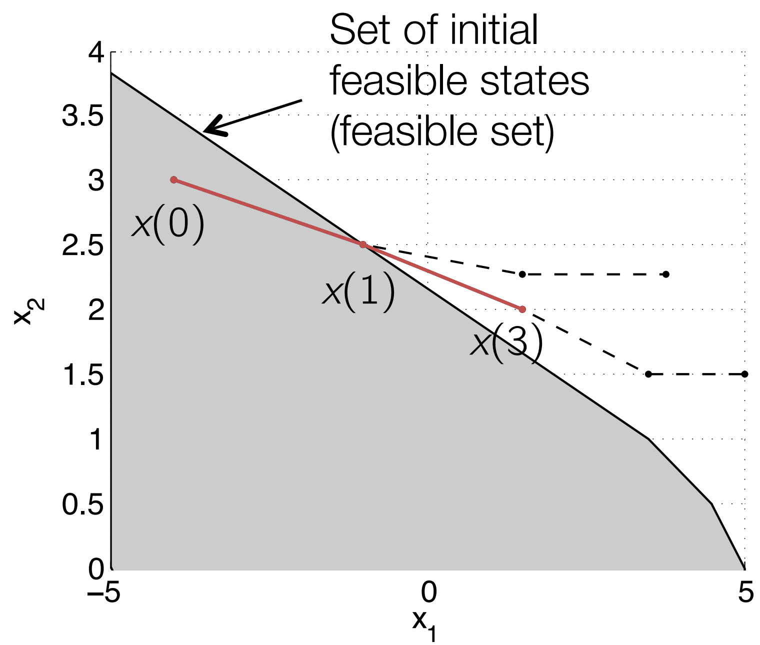
and the state constraints $\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

We choose $N = 3, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$.

Time step 1: $x(0)$=[-4 ; 4] , $u_0^*(x)$=-0.5

Time step 2: $x(1)$=[0 ; 3] , $u_0^*(x)$=-0.5

Time step 3: $x(2)$=[3 ; 2] MPC problem infeasible



Example: Loss of stability

Consider the unstable system: $x^+ = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

subject to the input constraints $-1 \le u \le 1$

and the state constraints $\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

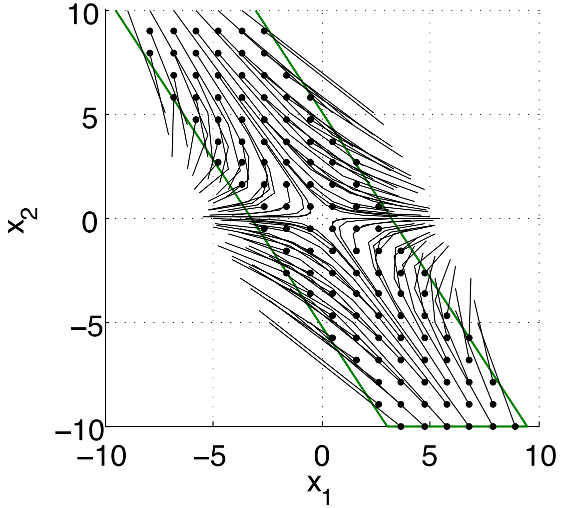
We choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and investigate the stability properties for different horizons N and weights R *by* solving the finite horizon MPC problem in a receding horizon fashion …

Example: Loss of stability

① R=10, N=2
② R=2, N=3
③ R=1 , N=4

* Initial points leading to trajectories that converge to the origin
• Intitial points that diverge

①


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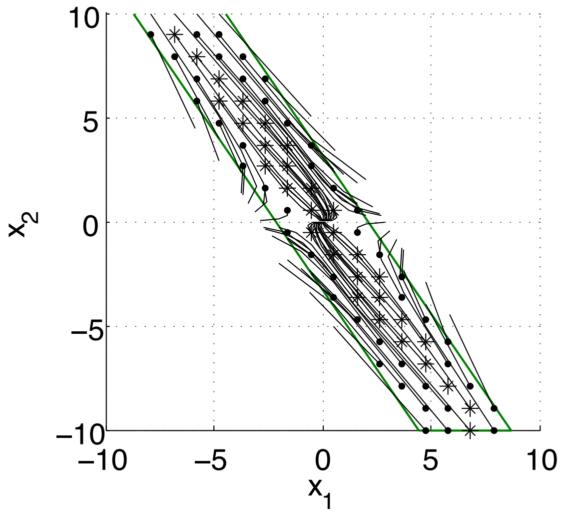

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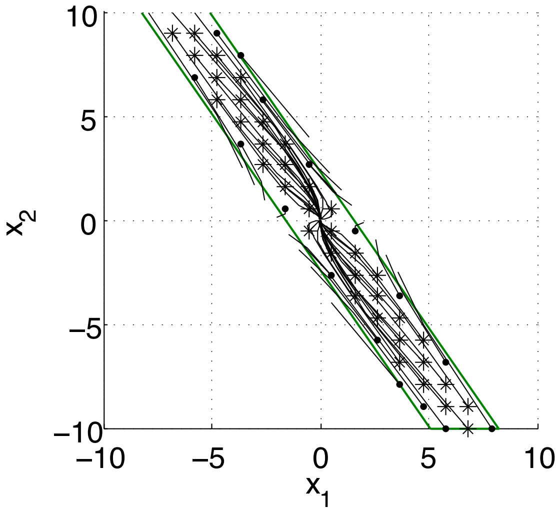


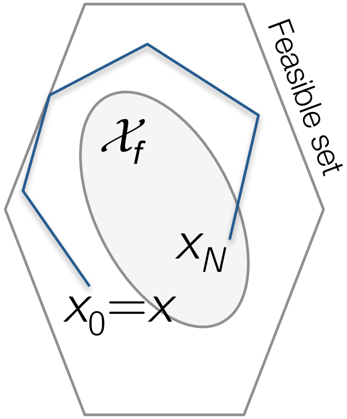
Parameters have complex effect on closed-loop trajectory

Feasibility and stability in MPC – Main Idea

Main idea:

Introduce terminal cost and constraints to explicitly ensure stability and feasibility:

$$
\begin{aligned}
J^*(x) = \min_{x,u} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N \quad \text{Terminal cost} \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& C x_i + D u_i \leq b \\
& x_N \in \mathcal{X}_f \quad \text{Terminal constraint} \\
& x_0 = x
\end{aligned}
$$

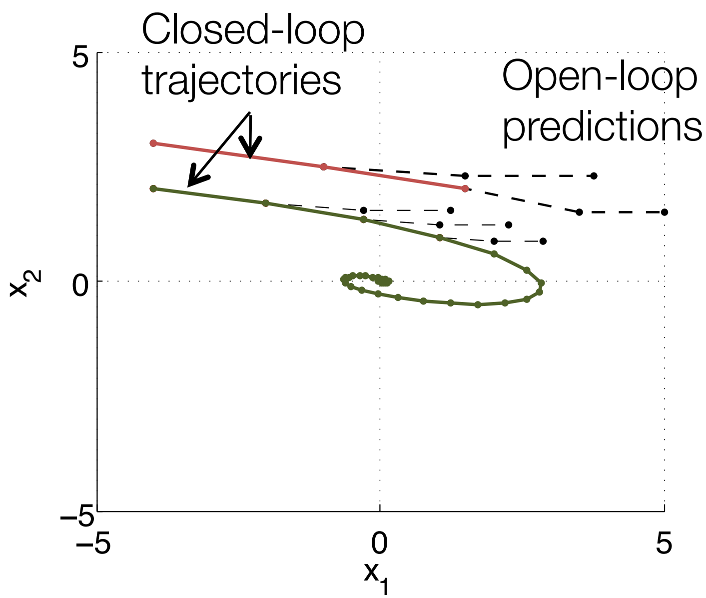


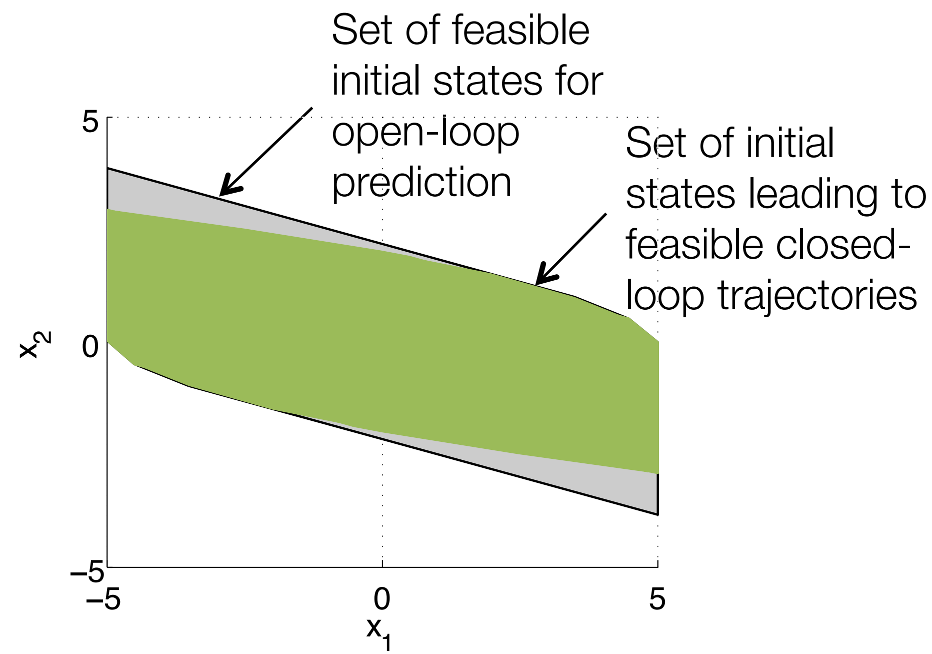
→ How to choose P and $\mathcal{X}_f$?

How to choose the terminal set and cost – Main Idea

- Problems originate from the use of a 'short sight' strategy

  → Finite horizon causes deviation between the open-loop prediction and the closed-loop system:





- Ideally we would solve the MPC problem with an infinite horizon, but that is computationally intractable

→ Design finite horizon problem such that it approximates the infinite horizon

How to choose the terminal cost

We can split the infinite horizon problem into two subproblems:

① Up to time $k=N$, where the constraints may be active

$$J^*(x) = \min_{x,u} \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i$$

$$\text{s.t.} \quad x_{i+1} = Ax_i + Bu_i$$

$$Cx_i + Du_i \leq b$$

$$x_0 = x$$

② For $k>N$, where there are no constraints active

$$+ \min_{x,u} \sum_{i=N}^{\infty} x_i^T Q x_i + u_i^T R u_i$$

$$\text{s.t.} \quad x_{i+1} = Ax_i + Bu_i \; ,$$

$$+ \; {x_N}^T P x_N$$

Unconstrained LQR starting from state x_N

- Bound the tail of the infinite horizon cost from N to ∞ using the **LQR control law** $u=K_{LQR}x$
- ${x_N}^T P x_N$ is the corresponding **infinite horizon cost**
- P is the solution of the discrete-time algebraic Riccati equation

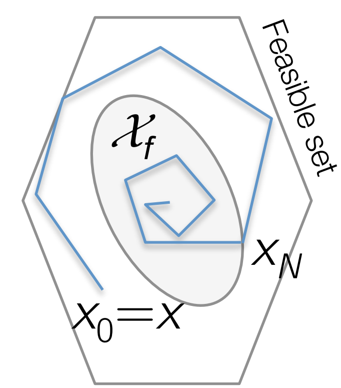
Choice of N such that constraint satisfaction is guaranteed?

How to choose the terminal set

Terminal constraint provides a sufficient condition for constraint satisfaction:

$$
\begin{aligned}
J^*(x) = \min_{x,u} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& C x_i + D u_i \leq b \\
& x_N \in \mathcal{X}_f \\
& x_0 = x
\end{aligned}
$$

Infinite horizon cost starting from x_N



- All input and state constraints are satisfied for the closed-loop system using the LQR control law for $x \in \mathcal{X}_f$
- Terminal set is often defined by linear or quadratic constraints

→ The bound holds in the terminal set and is used as a terminal cost
→ The terminal set defines the terminal constraint

In the following: Show that this problem setup provides feasibility and stability

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Formalize goals:
Definition of feasibility and stability

Goal 1: Feasibility at all times

Definition: Recursive feasibility

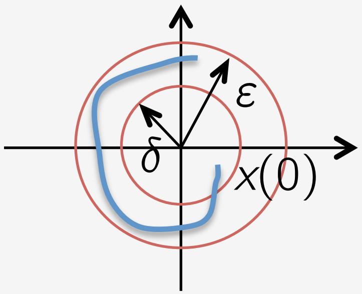
The MPC problem is called *recursively feasible*, if for all feasible initial states feasibility is guaranteed at every state along the closed-loop trajectory.

Goal 2: Stability

Definition: Lyapunov stability

The equilibrium point at the origin of system $x(k+1) = Ax(k) + B\kappa(x(k)) = f_\kappa(x(k))$ is said to be *(Lyapunov) stable* in $\mathcal{X}$ if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for every $x(0) \in \mathcal{X}$:

$$\|x(0)\| \leq \delta(\epsilon) \Rightarrow \|x(k)\| < \epsilon \; \forall k \in \mathbb{N} \ .$$



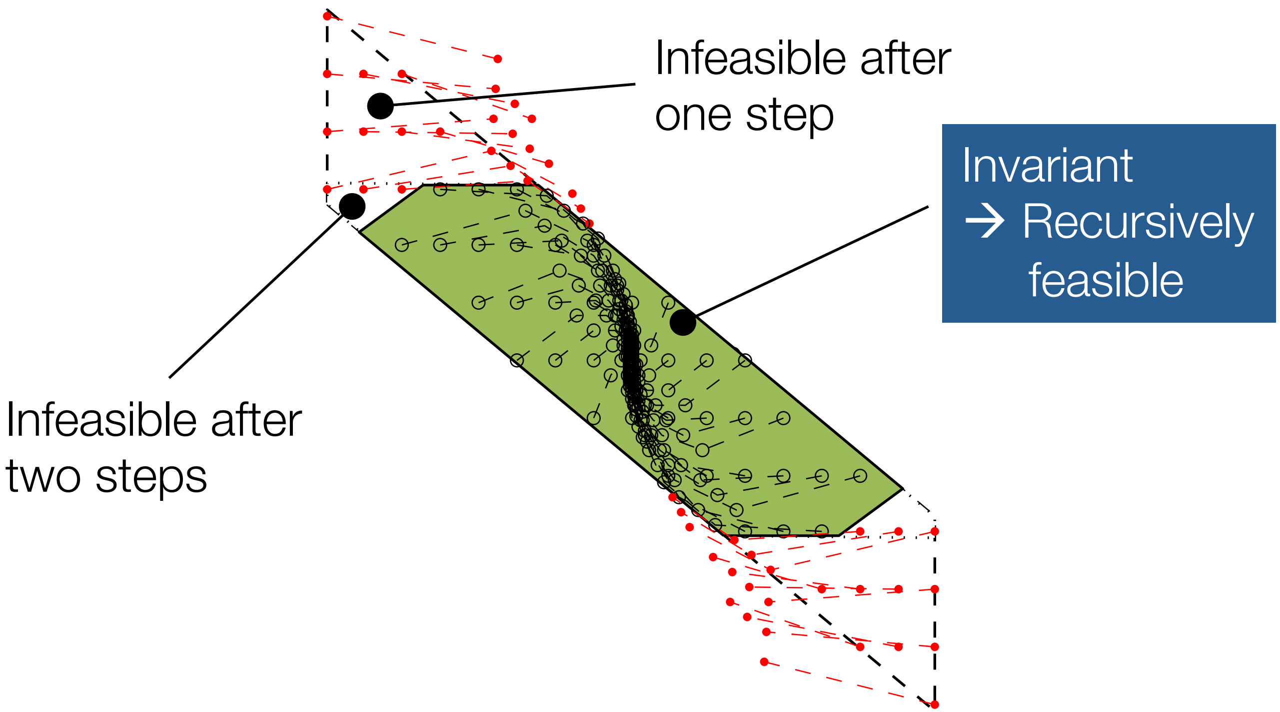
Employed concept for the analysis of feasibility: Invariant sets

Definition: Invariant set

A set $\mathcal{O}$ is called *positively invariant* for system $x(k+1) = f_\kappa(x(k))$, if

$$x(k) \in \mathcal{O} \Rightarrow f_\kappa(x(k)) \in \mathcal{O}, \quad \forall k \in \mathbb{N} \ .$$

The positively invariant set that contains every closed positively invariant set is called the *maximal positively invariant* set $\mathcal{O}_\infty$.



Analysis of Lyapunov stability

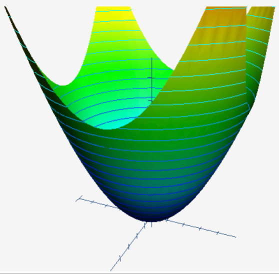
Lyapunov stability will be analyzed using Lyapunov's direct method:

Definition: Lyapunov function

Let $\mathcal{X}$ be a positively invariant set for system $x(k+1) = f_\kappa(x(k))$ containing a neighborhood of the origin in its interior. A function $V : \mathcal{X} \to \mathbb{R}_+$[1] is called a *Lyapunov function* in $\mathcal{X}$ if for all $x \in \mathcal{X}$:

$$V(x) > 0 \, \forall x \neq 0, \; V(0) = 0,$$

$$V(x(k+1)) - V(x(k)) \leq 0$$



Theorem: (e.g. *[Vidyasagar, 1993]*)

If a system admits a Lyapunov function in $\mathcal{X}$, then the equilibrium point at the origin is *(Lyapunov) stable* in $\mathcal{X}$.

[1] For simplicity it is assumed that $V(x)$ is continuous. This assumption can be relaxed by requiring an additional state dependent upper bound on $V(x)$, see e.g. [Rawlings & Mayne, 2009]

How to prove feasibility and stability of MPC

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point

NOTE: Recursive feasibility does not imply stability of the closed-loop system

- Prove stability by showing that the optimal cost function is a Lyapunov function

We will discuss two main cases in the following:

- Terminal constraint at zero: $x_N = 0$
- Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

For simplicity, we use the more general notation:

$$J(x) = \min_{\mathrm{x},\mathrm{u}} \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N)$$

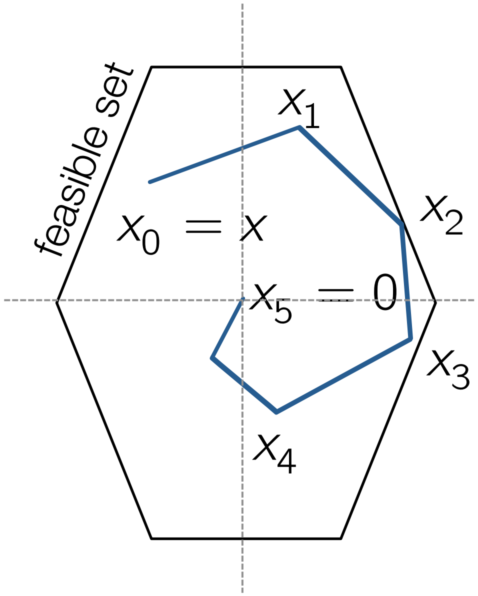
$l(x_i, u_i)$: stage cost; $V_f(x_N)$: terminal cost

(In the quadratic case: $l(x_i, u_i) = x_i^T Q x_i + u_i^T R u_i$, $V_f(x_N) = x_N^T P x_N$)

Stability of MPC – Zero terminal state constraint

Terminal constraint $x_N = 0$:

- Assume feasibility of x and let $[u_0^*, u_1^*, \ldots, u_{N-1}^*]$ be the optimal control sequence computed at x

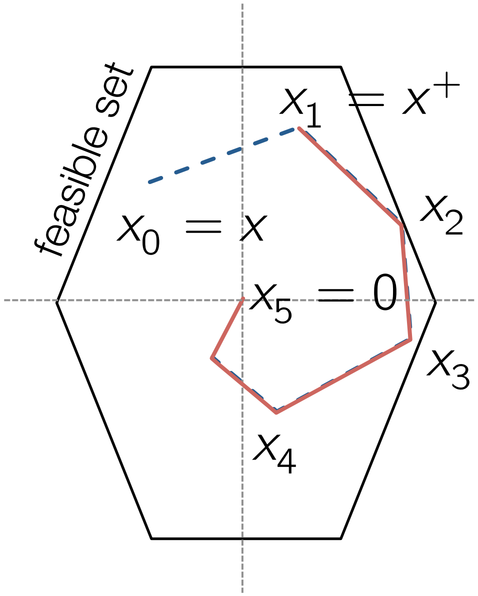


Stability of MPC – Zero terminal state constraint

Terminal constraint $x_N = 0$:

- Assume feasibility of x and let $[u_0^*, u_1^*, \ldots, u_{N-1}^*]$ be the optimal control sequence computed at x
- At x^+ the control sequence $[u_1^*, u_2^*, \ldots, u_{N-1}^*, 0]$ is feasible (apply 0 control input to stay at the origin)

→ **Recursive feasibility**



Stability of MPC – Zero terminal state constraint

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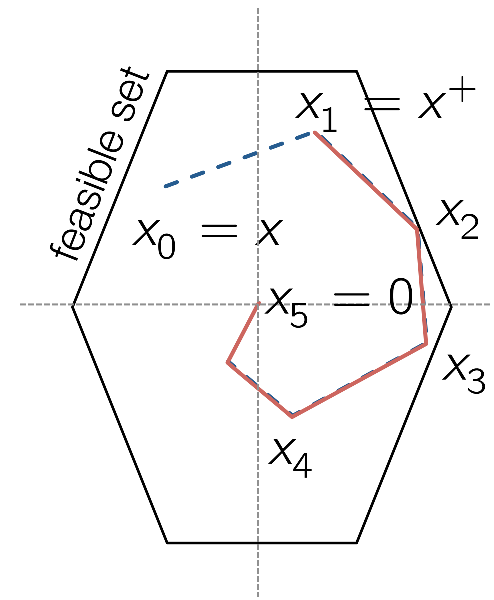
→ Recursive feasibility



- The associated cost function value is given by:

$$\tilde{J}(x^+) = J\ (x) - l(x, u_0) + l(0, 0)$$

Subtract cost at stage 0

Add cost for staying at 0

- We obtain for the optimal solution $J\ (x) \leq \tilde{J}(x)$

$$J\ (x^+) - J\ (x) \leq \tilde{J}(x^+) - J\ (x) \leq -l(x, u_0) \leq 0$$

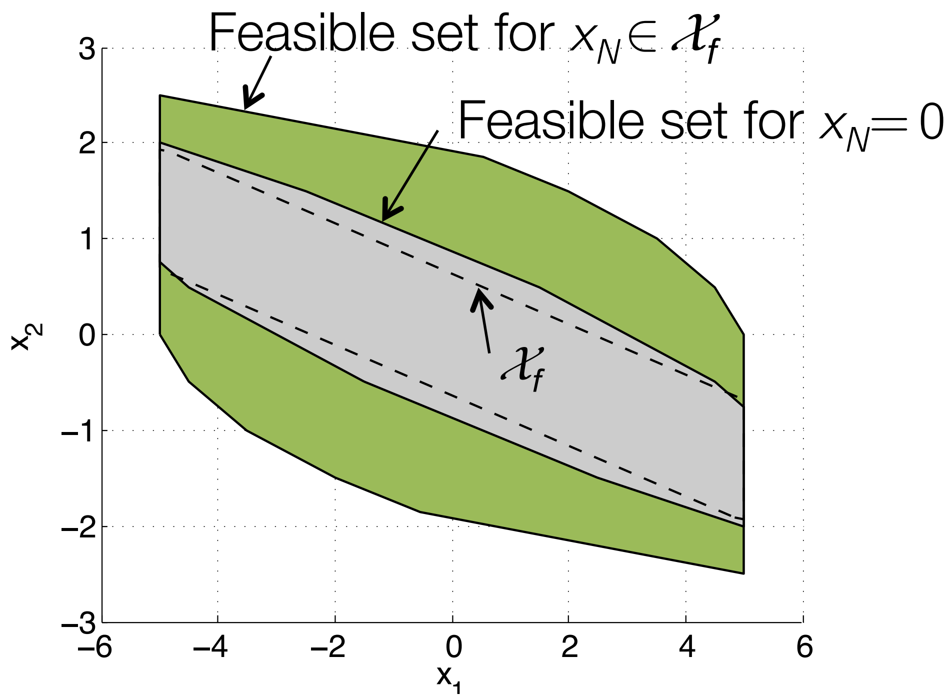
→ $J^*(x)$ is a Lyapunov function → (Lyapunov) Stability

Extension to more general terminal sets

Problem: The terminal constraint $x_N = 0$ reduces the feasible set

Goal: Use convex set for $\mathcal{X}_f$ that contains the origin in its interior

Example:



Double Integrator:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \le x \le \begin{bmatrix} 5 \\ 5 \end{bmatrix} \qquad -0.5 \le u \le 0.5$$

$$N = 5, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$$

How can the proof be generalized to the constraint $x_N \in \mathcal{X}_f$?

Stability of MPC – Main result

Consider that the following standard assumptions hold:

① The stage cost is a positive definite function, i.e. it is strictly positive and only zero at the origin

② The terminal set is invariant under the local control law $\kappa_f(x)$:

$$x^+ = Ax + B\kappa_f(x) \in \mathcal{X}_f \quad \text{for all } x \in \mathcal{X}_f$$

All state and input constraints are satisfied in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathbb{X},\ \kappa_f(x) \in \mathbb{U} \text{ for all } x \in \mathcal{X}_f$$

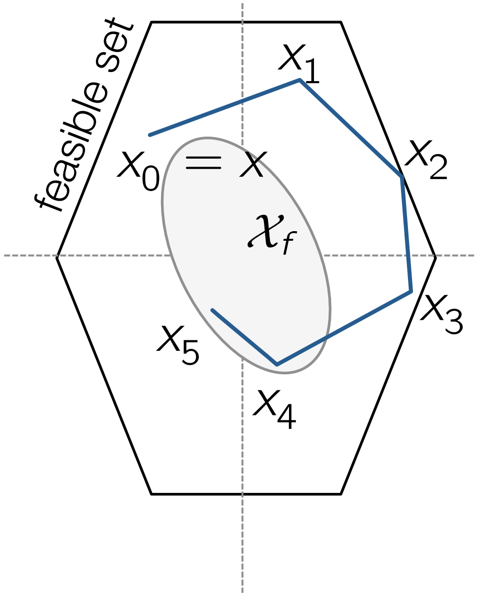
③ The terminal cost is a continuous Lyapunov function in the terminal set $\mathcal{X}_f$:

$$V_f(x^+) - V_f(x) \le -l(x, \kappa_f(x)) \text{ for all } x \in \mathcal{X}_f$$

→ The closed-loop system under the MPC control law is stable in the feasible set $\mathcal{X}_N$.

Stability of MPC – Outline of the proof

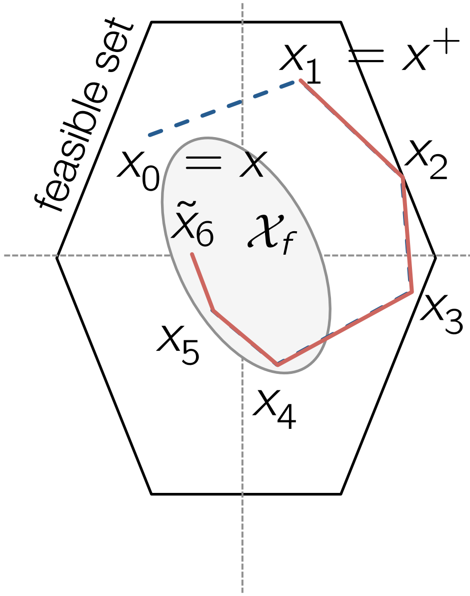
- Assume feasibility of x and let $[u_0^*, u_1^*, \ldots, u_{N-1}^*]$ be the optimal control sequence computed at x



Stability of MPC – Outline of the proof

- Assume feasibility of x and let $[u_0^*, u_1^*, \ldots, u_{N\text{-}1}^*]$ be the optimal control sequence computed at x
- At x^+ the control sequence $[u_1^*, u_2^*, \ldots, \kappa_f(x_N^*)]$ is feasible:
  x_N is in $\mathcal{X}_f$ → $\kappa_f(x_N^*)$ is feasible and $x_N^+ = Ax_N^* + B\,\kappa_f(x_N^*)$ *in* $\mathcal{X}_f$

→ Terminal constraint provides recursive feasibility



Stability of MPC – Outline of the proof

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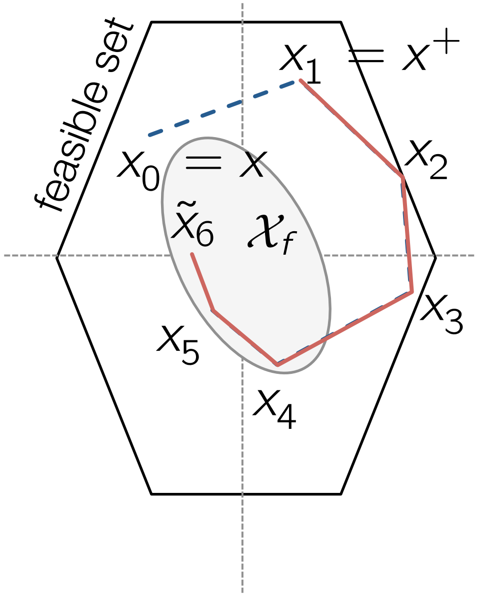


- The associated cost function value is given by:

$$\tilde{J}(x) = J^*(x) - l(x, u_0) + \underbrace{V_f(\tilde{x}_{N+1}) - V_f(x_N) + l(x_N, \kappa_f(x_N))}_{V_f(x) \text{ is a Lyapunov function: } \leq 0}$$

- We obtain for the optimal solution $J^*(x) \leq \tilde{J}(x)$:

$$J^*(x^+) - J^*(x) \leq \tilde{J}(x^+) - J^*(x) \leq -l(x, u_0) \leq 0$$

→ $J^*(x)$ is a Lyapunov function → (Lyapunov) Stability

Stability of MPC – Remarks

- The terminal set $\mathcal{X}_f$ and the terminal cost ensure recursive feasibility and stability of the closed-loop system.

  But: the terminal constraint usually reduces the region of attraction.

- If the open-loop system is stable, $\mathcal{X}_f$ can be chosen as the positively invariant set of the system under zero control input, which is feasible.

- Often no terminal set $\mathcal{X}_f$, but N is required to be sufficiently large to ensure recursive feasibility

  → Applied in practice, makes MPC work without terminal constraint

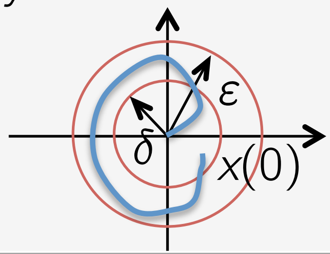
  → Determination of a sufficiently long horizon difficult

Proof of asymptotic stability

Definition: Asymptotic stability

Given a PI set $\mathcal{X}$ including the origin as an interior-point, the equilibrium point at the origin of system $x(k+1) = f_\kappa(x(k))$ is said to be *asymptotically stable* in $\mathcal{X}$ if it is

- *(Lyapunov) stable*
- *attractive in* $\mathcal{X}$, i.e. $\lim_{k\to\infty} \|x(k)\| = 0$ for all $x(0) \in \mathcal{X}$.



Extension of Lyapunov's direct method: (see e.g. *[Vidyasagar, 1993]*)

If the continuous Lyapunov function additionally satisfies

$$V(x(k+1)) - V(x(k)) < 0 \ \forall x \neq 0$$

then the closed loop system converges to the origin and is hence asymptotically stable.

Recall: Decrease of the optimal MPC cost was given by

$$J\ (x(k+1)) - J\ (x(k)) \leq -l(x(k), u_0)$$

where the stage cost was assumed to be positive and only 0 at 0.

→ The closed-loop system under the MPC control law is asymptotically stable.

Extension to nonlinear MPC

Consider the nonlinear system dynamics: $x^+ = f(x, u)$

→ Nonlinear MPC problem:

$$
\begin{aligned}
J^*(x) = \min_{\mathrm{x,u}} \quad & \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \\
\text{s.t.} \quad & x_{i+1} && = f(x_i, u_i) \\
& Cx_i + Du_i && \leq b \\
& x_N && \in \mathcal{X}_f \quad , \\
& x_0 && = x \quad ,
\end{aligned}
$$

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

→ Results can be directly extended to nonlinear systems.

Outline

- Motivating Example: Cessna Citation Aircraft
- Constraint satisfaction and stability in MPC
  - Main idea
  - Problem setup for the linear quadratic case
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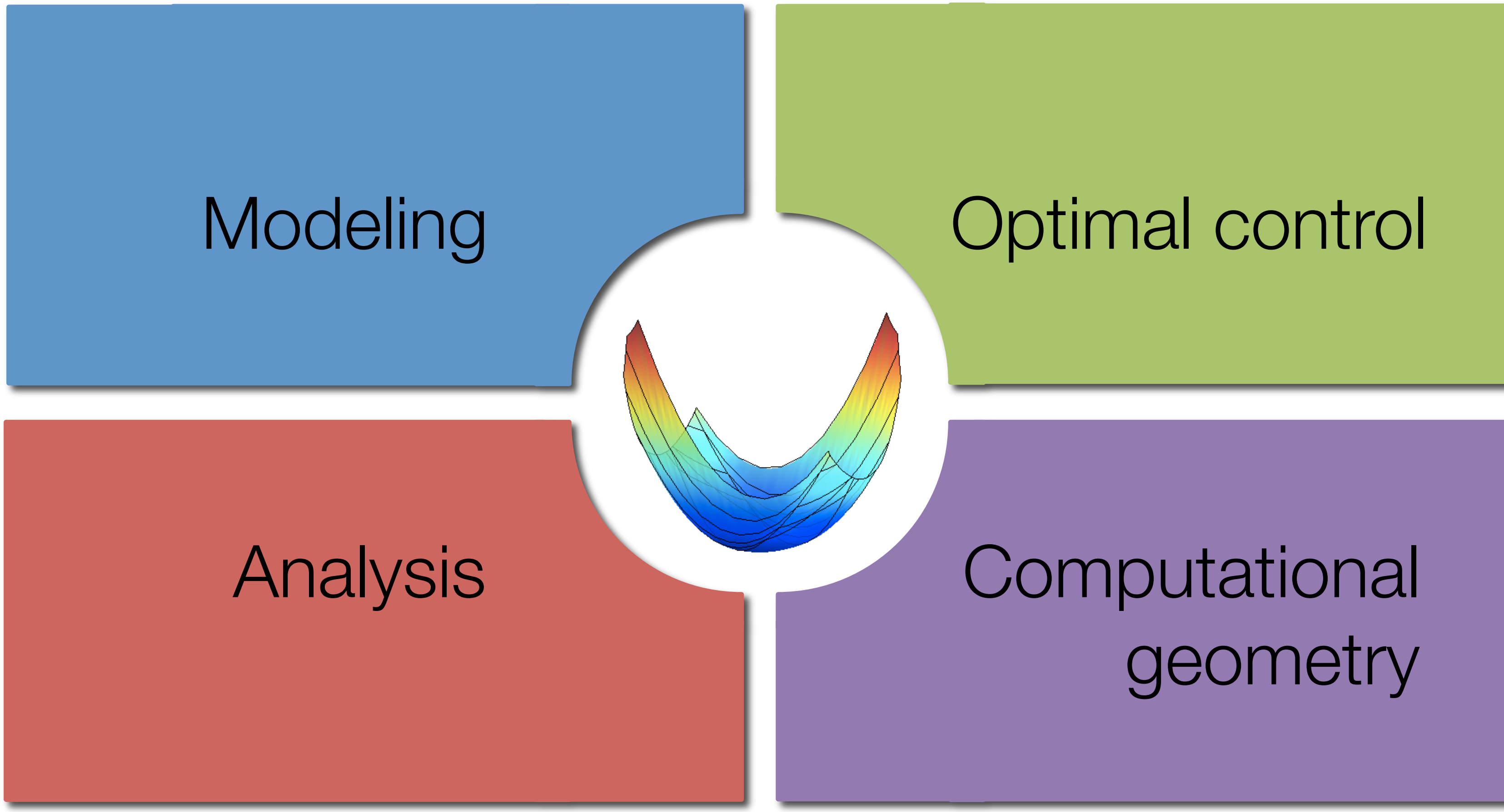
Multi-Parametric Toolbox



MPT Toolbox: M. Kvasnica, P. Grieder and M. Baotic

Real-Time Model Predictive Control via Multi-Parametric Programming: Theory and Tools
Michal Kvasnica
ISBN: 3639206444

Implementation using Matlab and MPT

1. Compute terminal weight P and LQR control law K by solving the discrete time Riccati equation

Matlab
```
[K,P]=dlqr(A,B,Q,R)
```

Double Integrator:

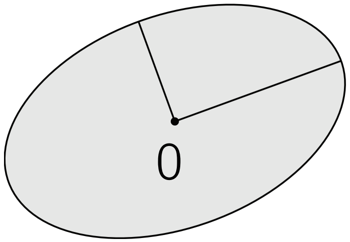
$$x^+ = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} x + \begin{bmatrix}0\\1\end{bmatrix} u$$

$$\begin{bmatrix}-5\\-5\end{bmatrix} \le x \le \begin{bmatrix}5\\5\end{bmatrix} \qquad -0.5 \le u \le 0.5$$

$$N = 5, Q = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, R = 10$$

2. Compute terminal set $\mathcal{X}_f$:
   - *Ellipsoidal invariant set* of the form

$$\mathcal{X}_f := \{x \,|\, x^T P_E x \le 1\}$$

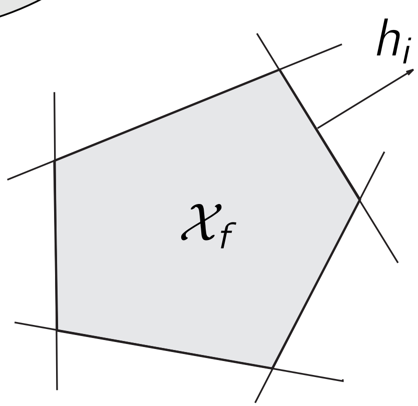


→ Can be written in the form of a Linear Matrix inequality (LMI)

[Boyd et al., LMIs in System and Control Theory, 1994*]*

   - *Polytopic invariant set* of the form

$$\mathcal{X}_f := \{x \,|\, Hx \le K\}$$



Implementation using Matlab and MPT: Polytopic Invariant Terminal Set

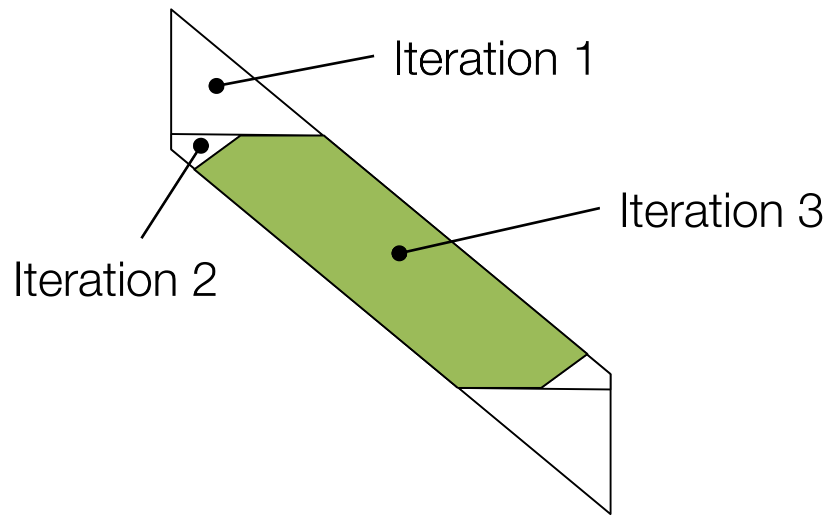
- Linear system $x^+ = Ax + Bu$
- LQR controller $u = Kx$
- Input and state constraints $U = \{u \mid u_{\min} \leq u \leq u_{\max}\}$ $X = \{x \mid x_{\min} \leq x \leq x_{\max}\}$

MPT
```
X = polytope([I;-I],[xmax; -xmin]) % State constraints
U = polytope([K;-K],[umax; -umin]) % Input constraints
```

- Compute maximum invariant set that satisfies the constraints

$$\mathcal{O}_\infty = \{x_k \mid (A+BK)^k x_k \in \mathcal{O}_\infty,\ \forall k \geq 0\}$$

MPT
```
P = X & U
while 1
  Pprev = P;
  P = P & inv(A+BK)*P;
  if Pprev == P, break; end
end
```



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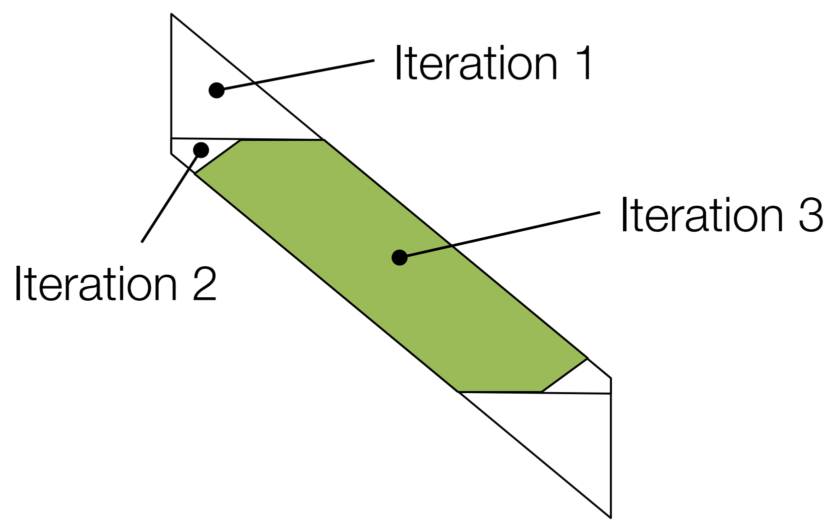


- Simpler procedure

MPT
```
X = mpt_infset(A+BK, X&U, 100);
```

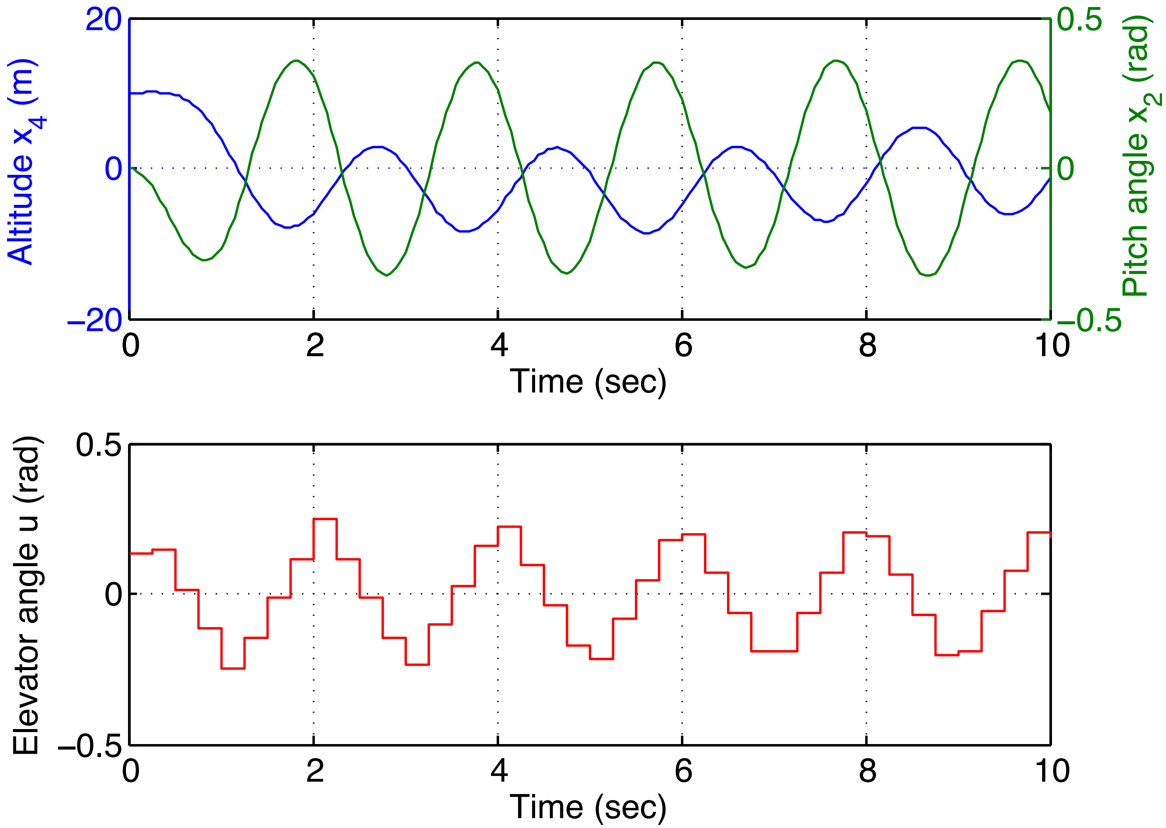
Example: Cessna Citation Aircraft Revisited

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$, $u_{-1} = u_{prev}$

Problem parameters:

Sampling time $T_s = 0.25 sec$,
$Q = I$, $R = 10$, $N = 4$

Decrease in the prediction horizon causes loss of the stability properties



Example: Cessna Citation Aircraft
Terminal cost and constraint provide stability guarantee

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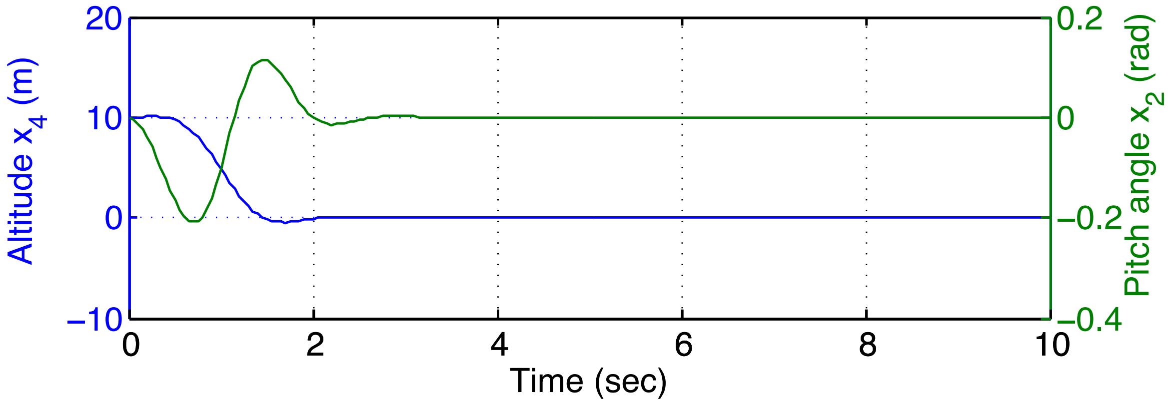
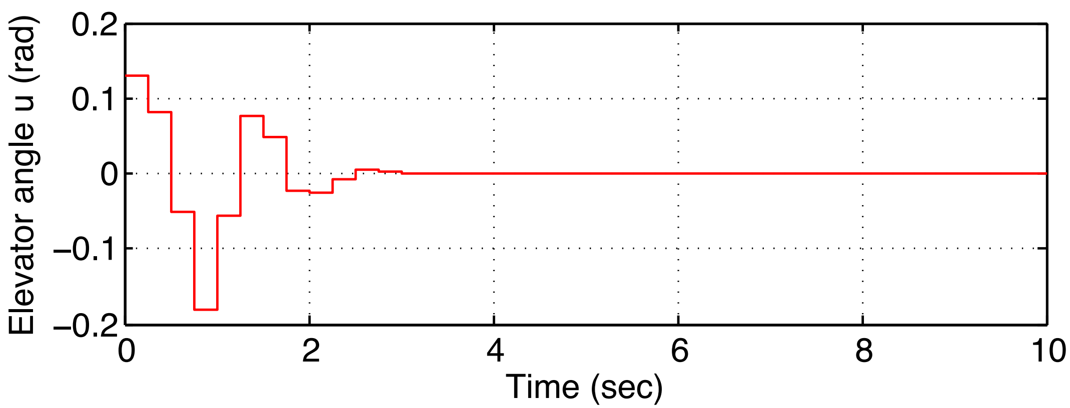
Problem parameters:

Sampling time $T_s = 0.25sec$,
$Q = I$, $R = 10$, $N = 4$



→ Inclusion of terminal cost and constraint provides stability



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  - Reference Tracking
  - Uncertain Systems
  - Soft constrained MPC

Extensions
– Reference Tracking

Consider the system
$$x^+ = Ax + Bu$$
$$y = Cx$$

Regulation vs. Tracking:

- **Regulation:** Reject disturbances around one desirable steady-state
- **Tracking:** Make output follow a given reference r

Steady-state computation:
Compute steady-state that yields the desired output

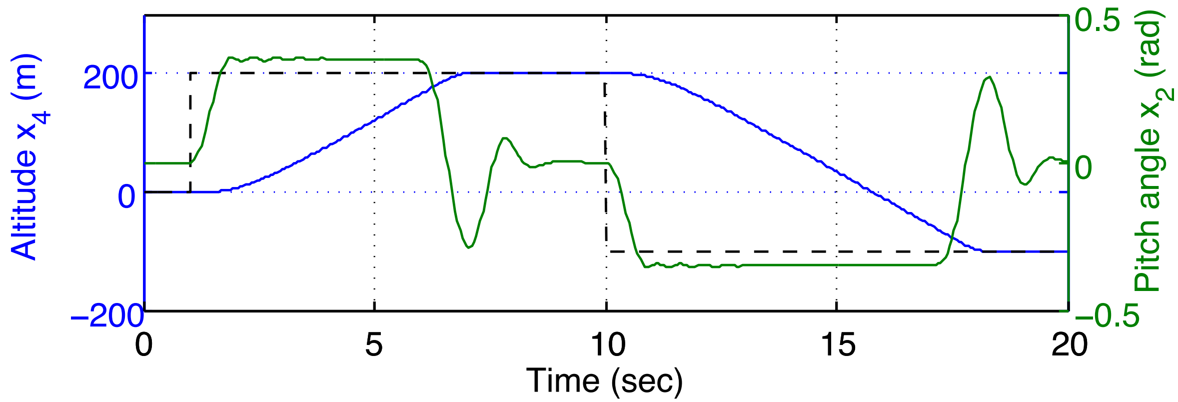
$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

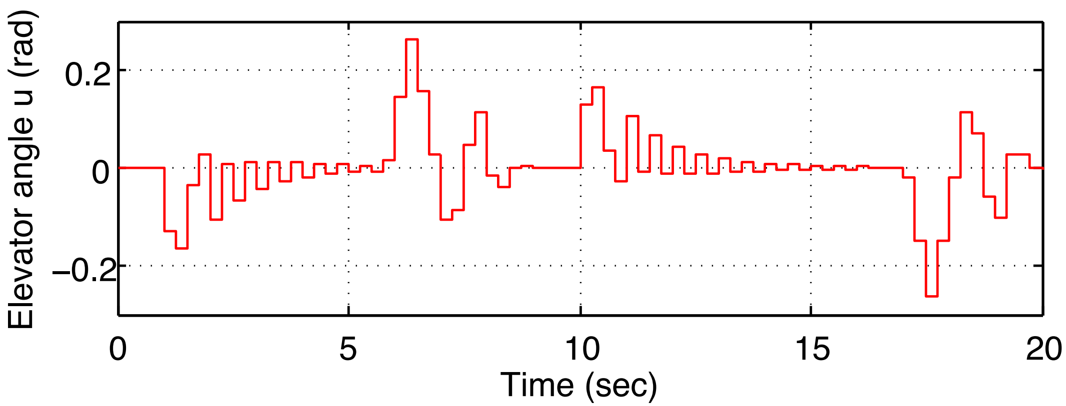
Note: Many solutions possible

For a solution to exist the matrix must be full column rank and the steady-state must be feasible, i.e. $x_{ss} \in \mathbb{X}$, $u_{ss} \in \mathbb{U}$

Example: Cessna Citation Aircraft
Reference tracking

Track given altitude profile:





Problem parameters:

Sampling time T_s=0.25*sec*,
$Q=I$, $R=10$, $N=10$

Extensions – Reference Tracking

MPC problem for reference tracking:

Penalize the deviation from the desired steady-state

$$J^*(x) = \min_{x,u} \sum_{i=0}^{N-1} (x_i - x_{ss})^T Q (x_i - x_{ss}) + (u_i - u_{ss})^T R (u_i - u_{ss}) + (x_N - x_{ss})^T P (x_N - x_{ss})$$

$$\begin{aligned} \text{s.t.} \quad & x_{i+1} && = Ax_i + Bu_i \\ & Cx_i + Du_i && \leq b \\ & x_0 && = x \end{aligned}$$

Here: Reference assumed constant over horizon.
If reference trajectory is known this can be included → Preview

- Steady-state is computed at each time step
- Same problem structure as the standard MPC problem with additional parameter r

Steady-state offset: If the model is not accurate the output will show an offset from the desired reference

Note: *Parameterized* terminal set required for stability.

Extensions
– Offset-free reference tracking

Goal: Zero steady-state tracking error, i.e. $y(k) - r(k) \to 0$ for $k \to \infty$

Consider the augmented model:
$$x^+ = Ax + Bu + B_w w$$
$$y = Cx + C_w w$$
with $w \in \mathbb{R}^w$.

Assume size of w=number of states , C_w=I and $det \begin{bmatrix} A - I & B_w \\ C & I \end{bmatrix} \neq 0$.
Then the augmented system is observable.

→ We can correct for the disturbance by choosing x_{ss}, u_{ss} such that

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} -B_w \hat{w} \\ r - C_w \hat{w} \end{bmatrix}$$

where $\hat{w}$ is an estimate of the uncertainty obtained from a state observer.

→ See *[Mäder et al., Automatica 2009]* for more details.

Extensions
– Uncertain systems

In practice, the nominal system model will not be accurate due to *model mismatch* or *external disturbances* that are acting on the system.

Consider the uncertain system

$$x^+ = Ax + Bu + B_w w$$

where $w \in \mathcal{W}$ is a bounded disturbance.

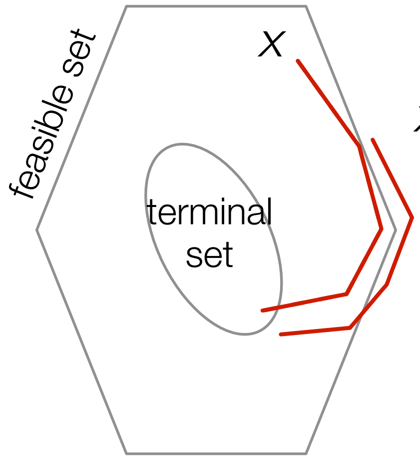
Stability:

- In the presence of uncertainties, asymptotic stability of the origin can often not be achieved. Instead, it can be shown that the trajectories converge to a neighborhood of the origin.
- Stability can be analyzed in the framework of **input-to-state stability.**

  Idea: Effect of the uncertainty on the system is bounded and depends on the size of the uncertainty.

Feasibility?

Extensions
– Approaches for uncertain systems



$$x^{+} = Ax + Bu + B_w w$$

Trajectories differ from what is expected using the nominal system model
→ Uncertainties can lead to infeasibility of the MPC problem

Soft constrained MPC:

Idea: Tolerate temporary violation of state constraints by constraint relaxation

→ Feasibility of the optimization problem despite disturbances

Robust MPC:

Idea: Design MPC problem for the worst-case disturbance by constraint tightening

→ Constraint satisfaction and stability of the uncertain system

Extensions
– Soft constrained MPC

Idea: Input constraints are hard (e.g. actuator limitations),
state constraints may (temporarily) be violated

- Introduce soft state and terminal constraints by means of slack variables
- Introduce penalties on the slack variables in the cost

$$
\begin{aligned}
J^*(x) = \min_{x,u} \quad & \sum_{i=0}^{N-1} l(x_i, u_i) + l_\epsilon(\epsilon_i) + V_f(x_N) + l_\epsilon(\epsilon_N) \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& Cx_i + Du_i \leq b + \epsilon_i \\
& G_N x_N \leq f_N + \epsilon_N \\
& x_0 = x
\end{aligned}
$$

Penalties on the amount of constraint violation

Slack variables

Standard soft constrained MPC setup does not provide stability guarantees

→ New soft constrained MPC setup with (robust) stability properties developed in *[Zeilinger et al., CDC 2010]*

Further extensions

- Reference tracking with recursive feasibility guarantees

  [Gilbert et al., 1994; Bemporad 1998; Gilbert & Kolmanovsky 1999, 2002; Limon et al., 2008; Borrelli et al., 2009; Ferramosca et al., 2009; …]

- Move blocking: Reduce the computational complexity by fixing the inputs or its derivatives to be constant over several time steps

  [Li & Xi 2007, 2009; Cagienard et al. 2007; Gondhalekar & Imura, 2010; …]

- Stochastic MPC: Consider uncertainties that are unbounded and/or follow a certain distribution
  - Probabilistic constraints
  - Expected value constraints
  - Expected value cost

  [Lee et al. 1998; Couchman et al., 2005; Cannon et al., 2007; Grancharova et al. 2007; Hokayem et al., 2009; Cinquemani et al., 2011; …]