# Fixed Complexity Explicit Model Predictive Control

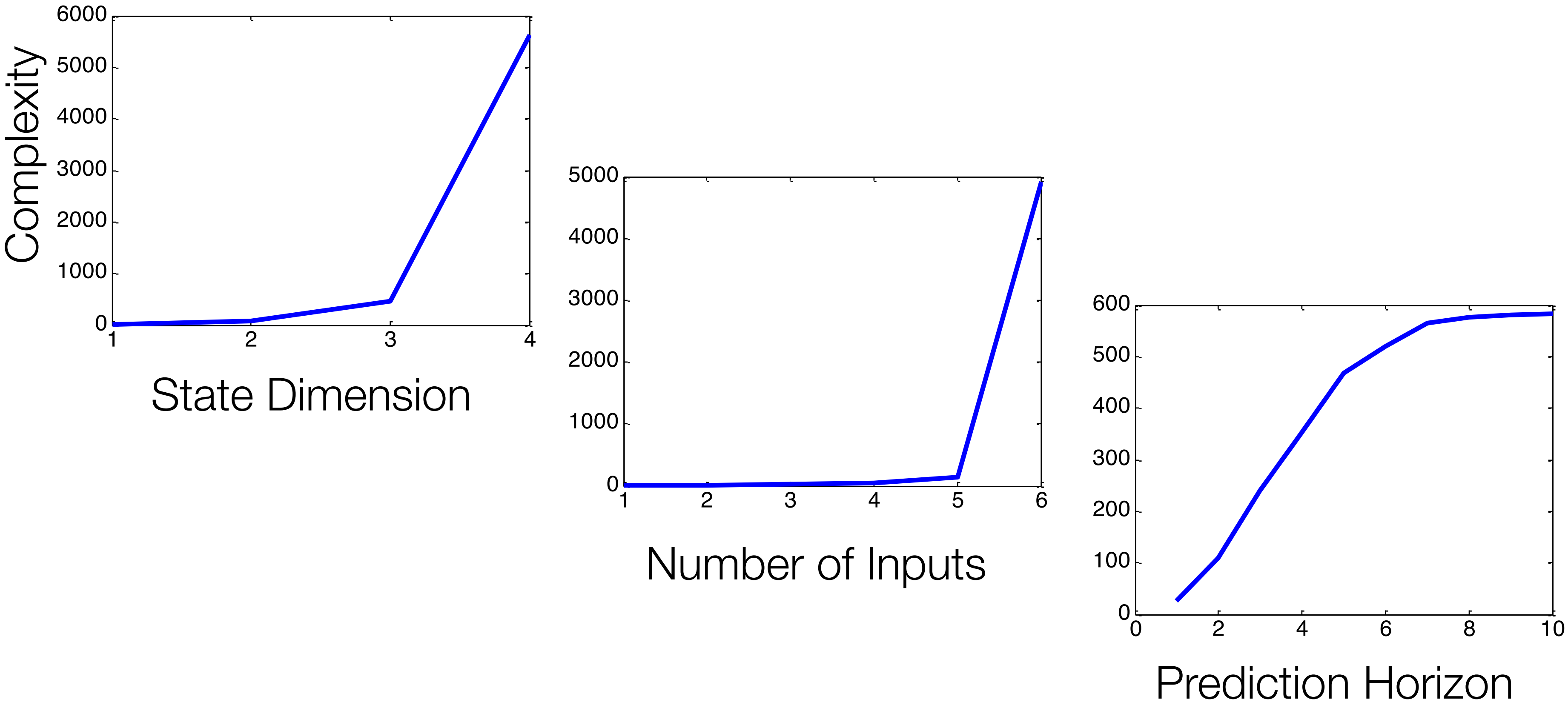
Colin Jones and Melanie Zeilinger

Automatic Control Laboratory, EPFL

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

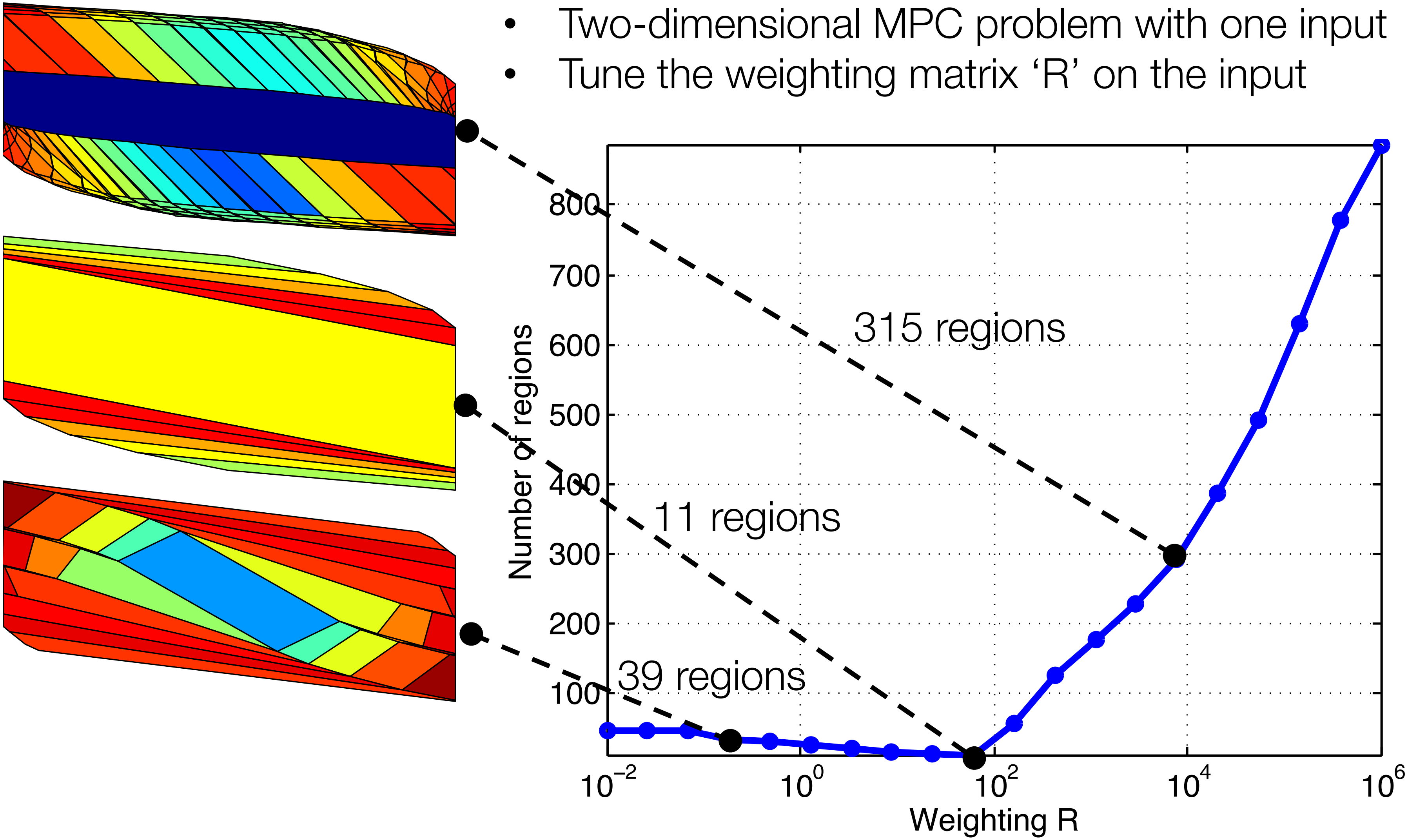
# Complexity highly sensitive to problem size

Rapid increase in complexity with parameters



# Impact of Tuning is Highly Uncertain

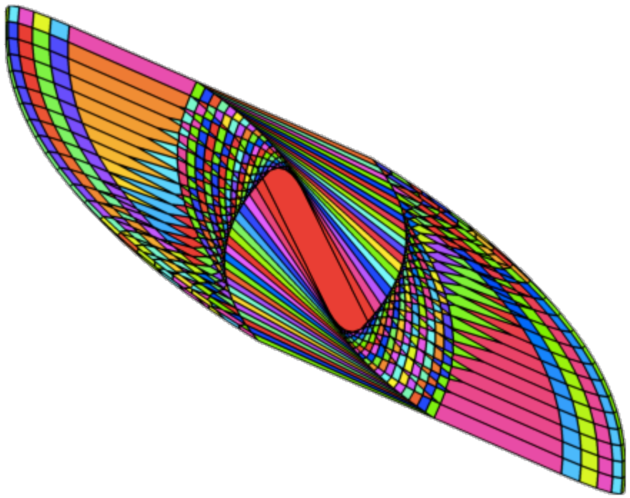
- Two-dimensional MPC problem with one input
- Tune the weighting matrix ‘R’ on the input



# Limiting Factor: Complexity

Number of regions determine important properties

- online computation time
- storage requirements
- offline processing time



Complexity is a property of the problem

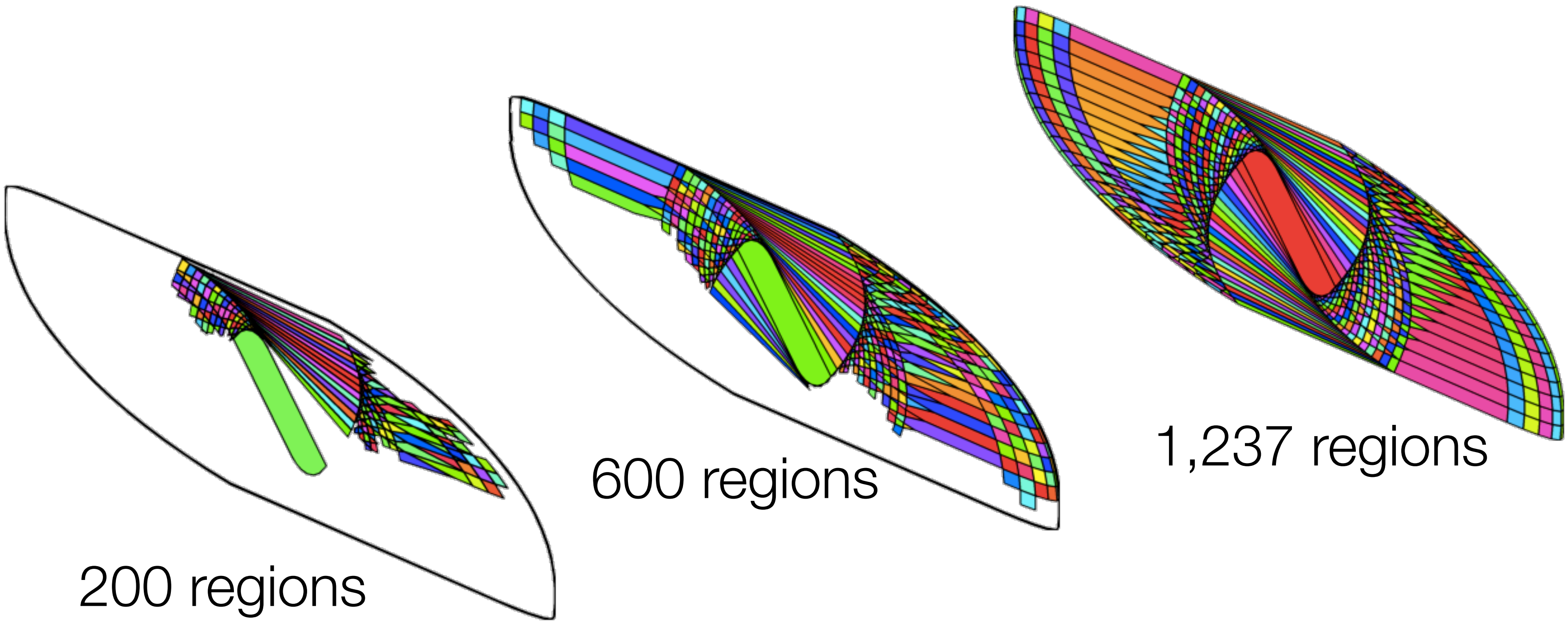
- Nothing is known about relationship between problem parameters and complexity

**Fixed-complexity explicit MPC: Sub-optimal controller of exactly *M* regions**

- Is it stable?
- Invariant?
- What level of sub-optimality?

# Current optimal algorithms

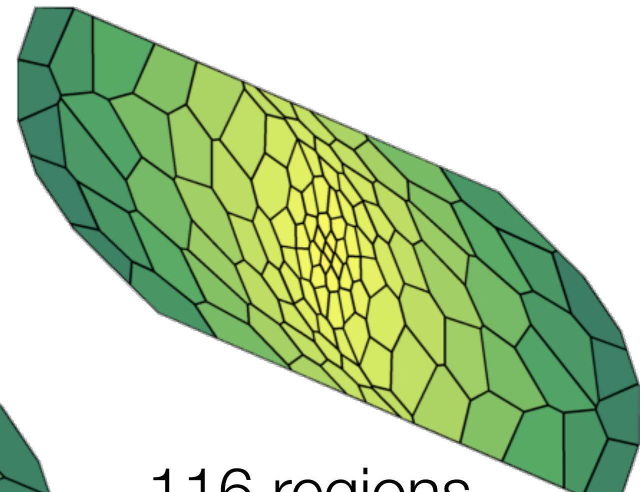
- Hardware can only store *M* regions
- Current algorithms are non-incremental
  - output meaningless until complete
  - cannot stop early: *M* regions gives nothing



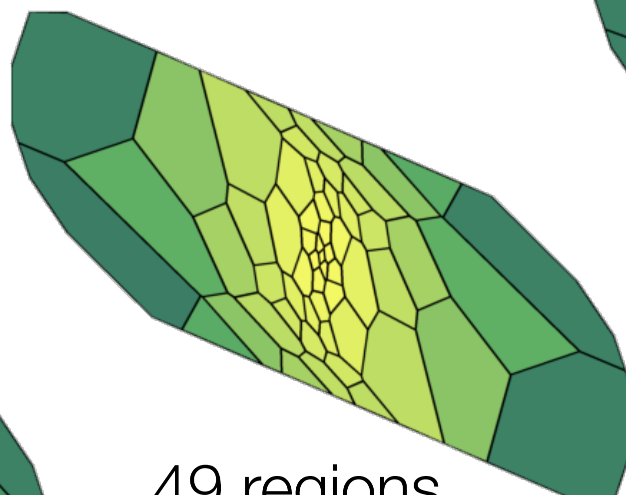
# Goal: Incremental algorithms

Assume: Embedded processor can store *M* regions

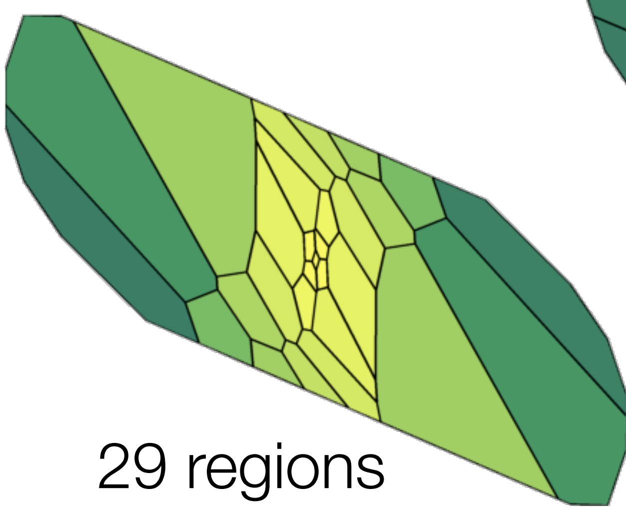
- Initially poor approximation
- Improve performance by adding complexity
- Stop when *M* regions reached
  - Certificate of stability/performance



116 regions



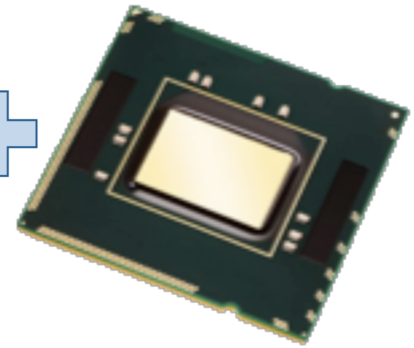
49 regions

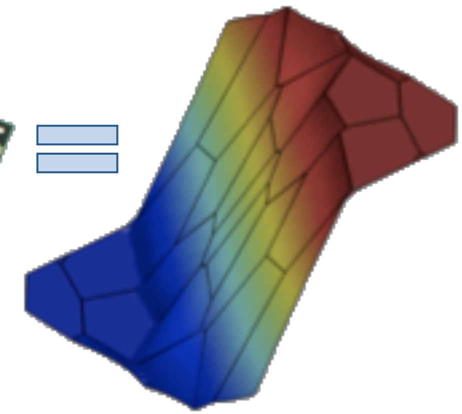


29 regions

# Real-time synthesis : Complexity as a specification

$$u^\star(x) = \operatorname*{argmin}_{u_i} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$$
$$x_N \in \mathcal{X}_N$$
$$x_0 = x$$



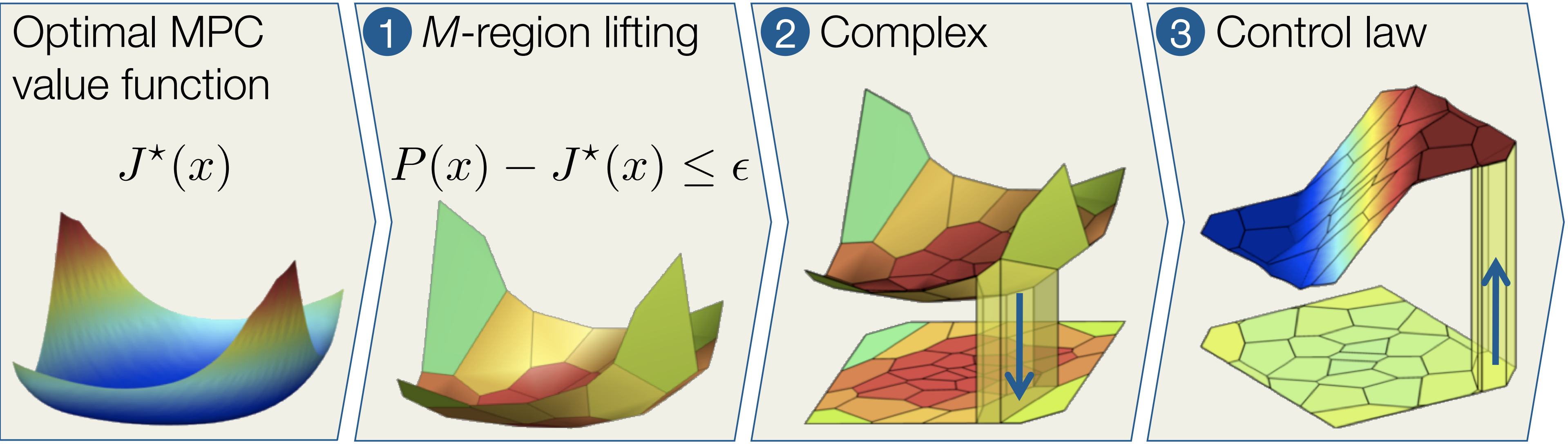


Properties of fixed-complexity MPC:

- Tradeoff between complexity and optimality
  - Real-time synthesis
  - Control extremely high-speed systems
- Process any convex MPC problem
- Synthesis of control law to software is verifiable

*[Jones, Barić, Morari, 2007]*
*[Jones, Morari, 2010]*
*[Summers, Jones, Lygeros, Morari 2010]*

# Real-time explicit MPC : Offline processing




Given optimal controller:

1. Compute convex polyhedral function of *M* facets
2. Define complex as projection of lifting facets
3. Interpolate optimal control law at vertices of complex

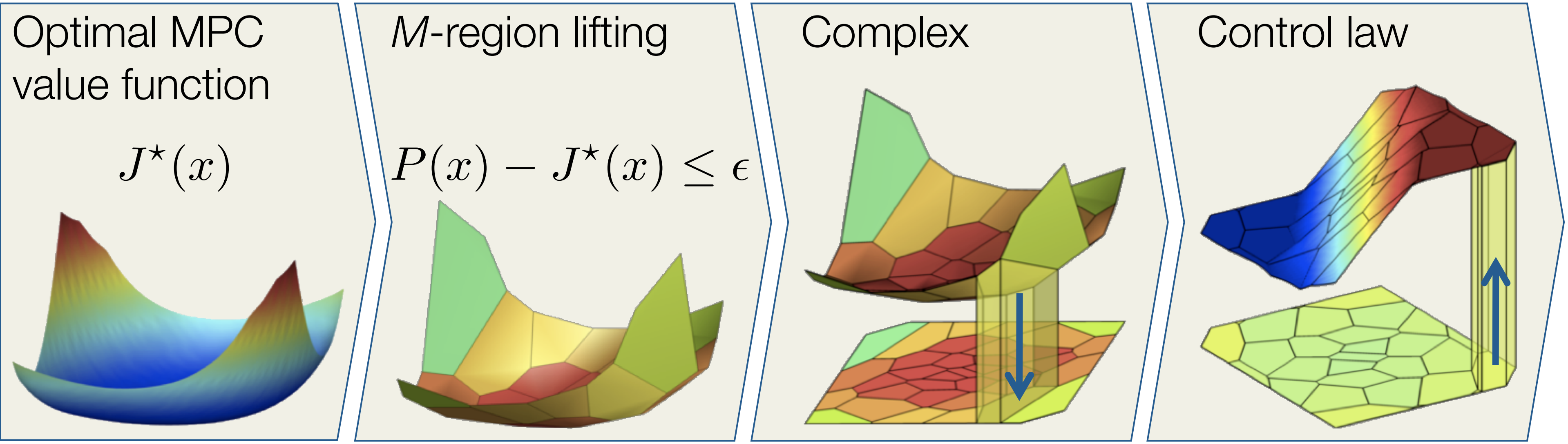
$$
\begin{aligned}
J^\star(x_0) = \min_{u_i}\ & V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{s.t. } & x_{i+1} = f(x_i, u_i) \\
& (x_i, u_i) \in \mathcal{X} \times \mathcal{U} \\
& x_N \in \mathcal{X}_N
\end{aligned}
$$

Result : Piecewise polynomial controller of *M* regions



[C.N. Jones and M. Morari, TAC 2010]

# Real-time explicit MPC : Properties




Real-time explicit MPC:

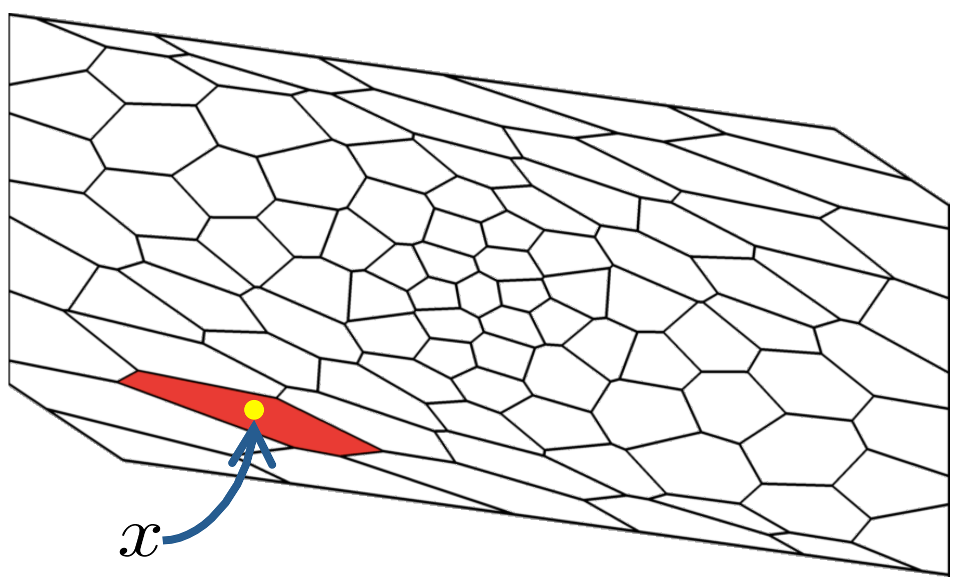
- Is computable in micro- to nanoseconds
- Satisfies constraints
- Stabilizes the system
- Complexity/performance tradeoff

# Computational bottleneck : Point Location

Goal : General class of functions that can be evaluated in µs / ns

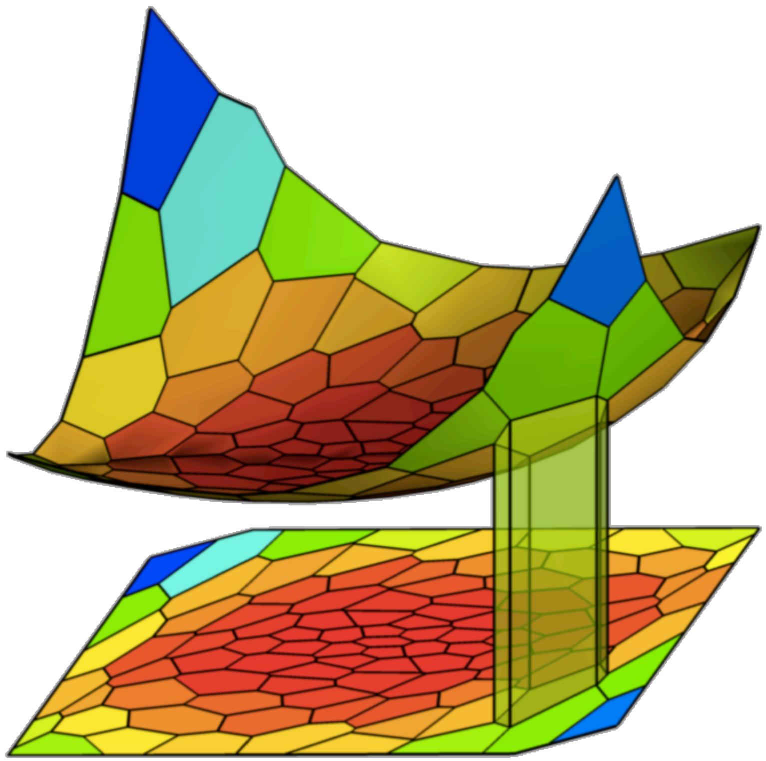
Critical operation : point location

Given $x \in \mathbb{R}^n$ and cell complex $\{C_1, \ldots, C_m\}$ find $i$ such that $x \in C_i$



# Liftable complexes are log-time computable

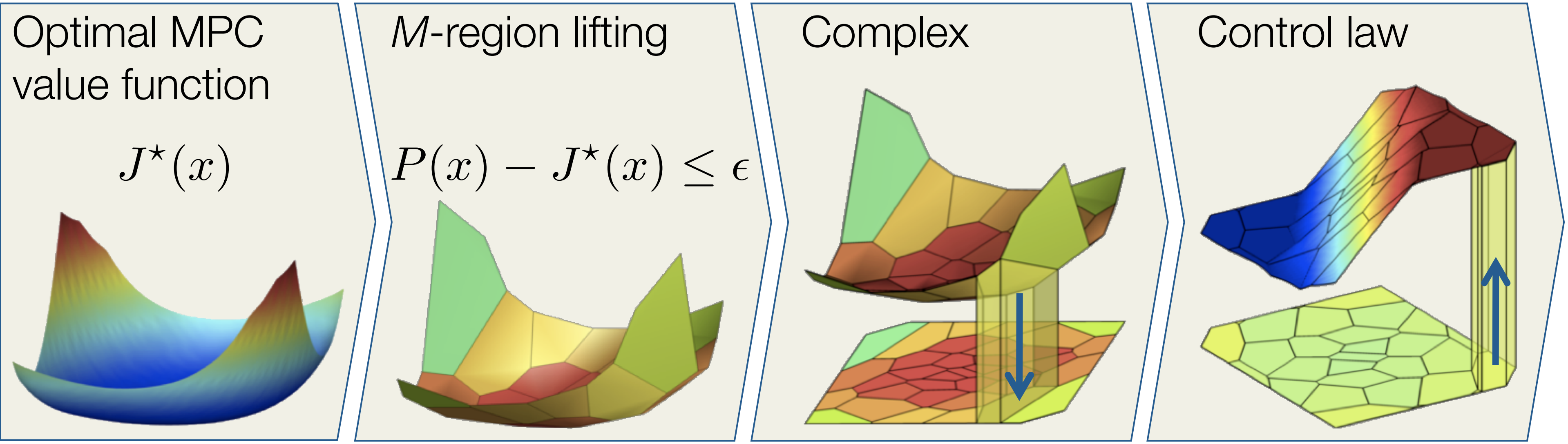
Goal : General class of functions that can be evaluated in μs / ns



- A polytopic complex is a power diagram iff it has a convex lifting *[Aurenhammer, 1991]*
- Power diagram point location can be done in O(log n) *[Mount et al, 1998]*
- Explicit MPC with PWA cost has a lifting, but quadratic does not *[Jones et al, 2006]*

Result : Design convex lifting ⇒ Log-time evaluation

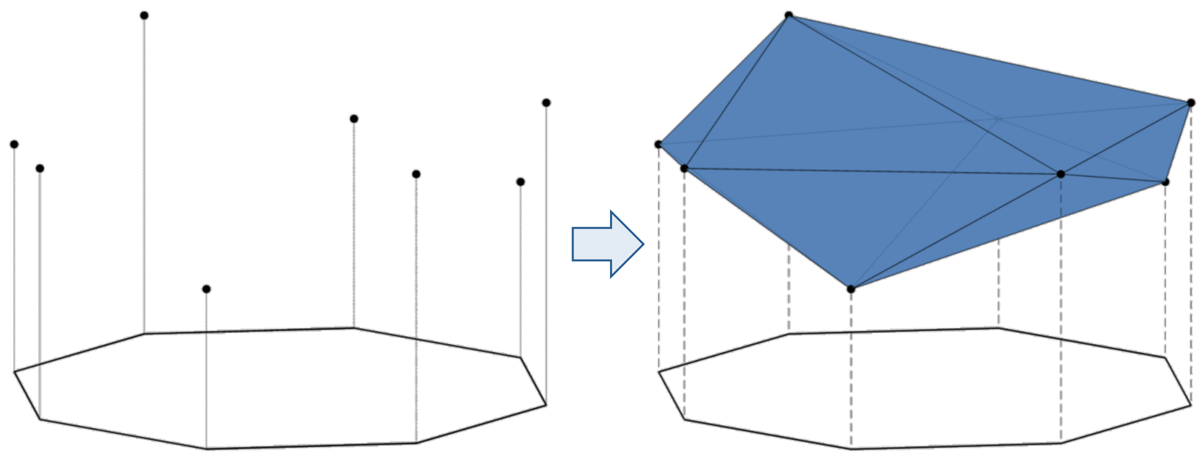
# Real-time explicit MPC : Properties




Real-time explicit MPC:

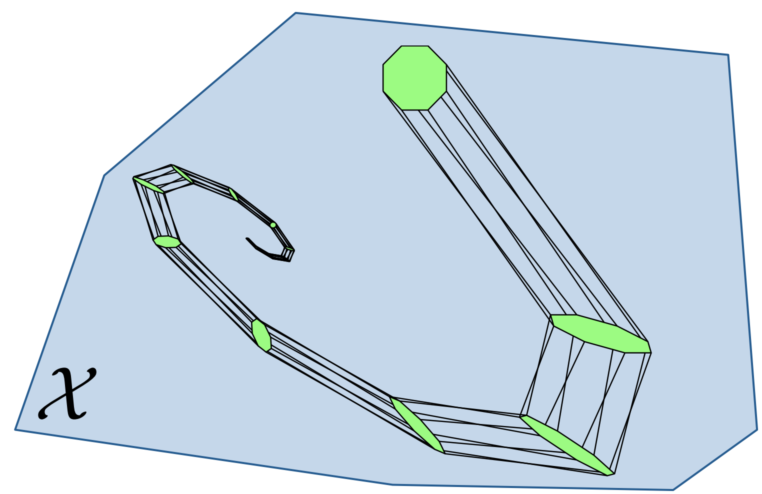
- Is computable in micro- to nanoseconds <= Lifting function
- **Satisfies constraints**
- Stabilizes the system
- Complexity/performance tradeoff

# Constraint satisfaction : Barycentric interpolation



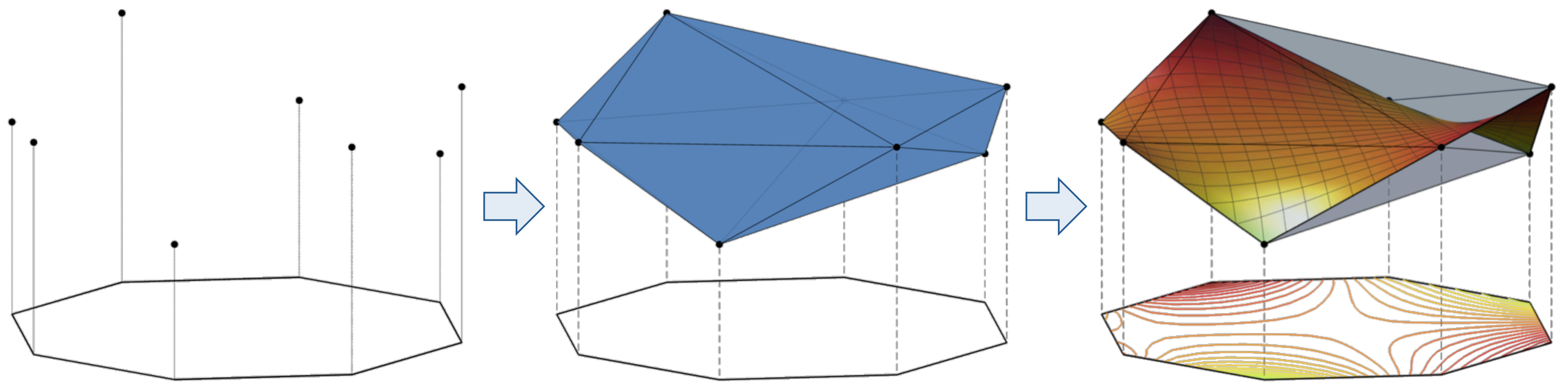
Goal : Interpolate vertices and satisfy constraints

- Convexity : Any interpolation inside convex hull is feasible




**Result : Vertices feasible ⇒ Convex hull feasible**

# Constraint satisfaction : Barycentric interpolation



Goal : Interpolate vertices and satisfy constraints

- Convexity : Any interpolation inside convex hull is feasible

⇒ Barycentric interpolation

⇒ How to define *w*?

$$w_v(x) \geq 0 \qquad \text{positivity}$$

$$\sum_{v \in \text{extreme}(S)} w_v(x) = 1 \qquad \text{partition of unity}$$

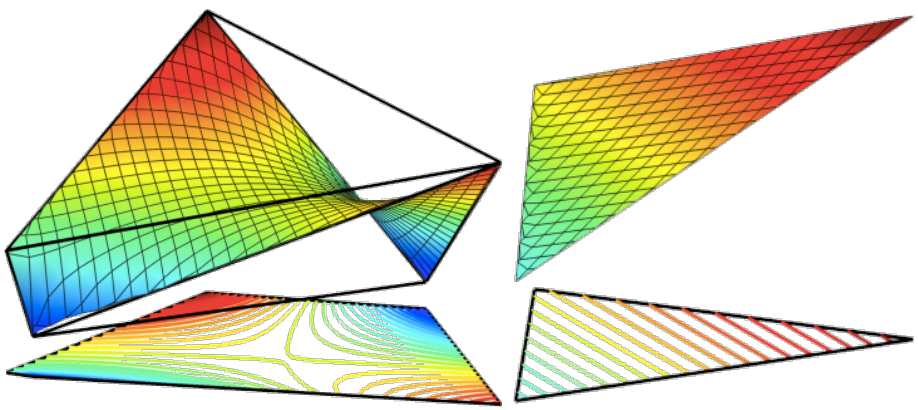
$$\sum_{v \in \text{extreme}(S)} v w_v(x) = x \qquad \text{linear precision}$$

# Constraint satisfaction : Barycentric interpolation

**Thm:** $$\tilde{u}(x) = \sum_{v \in V} \frac{u^\star(v)\alpha_v}{\|v - x\|_2}$$ is barycentric for $\operatorname{conv}(V)$

- $\alpha_v$ : area of facet $v$ in dual polytope (pre-computed)

- Valid for *any polytope*
- Low data storage
- Evaluation in µs

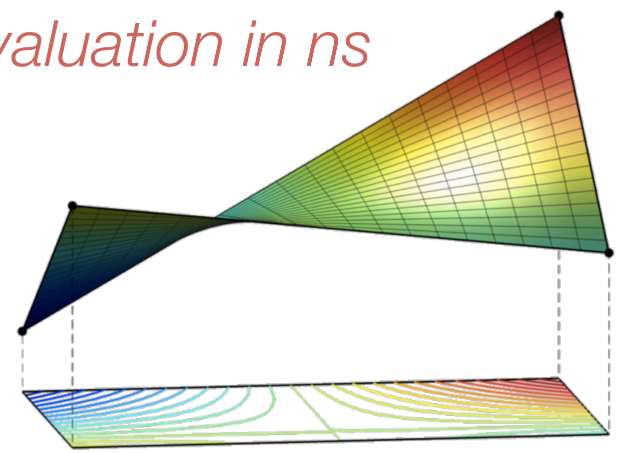


[Schaefer et al, 2008]

**Thm:** Tensor-product expansion of second-order interpolants is barycentric

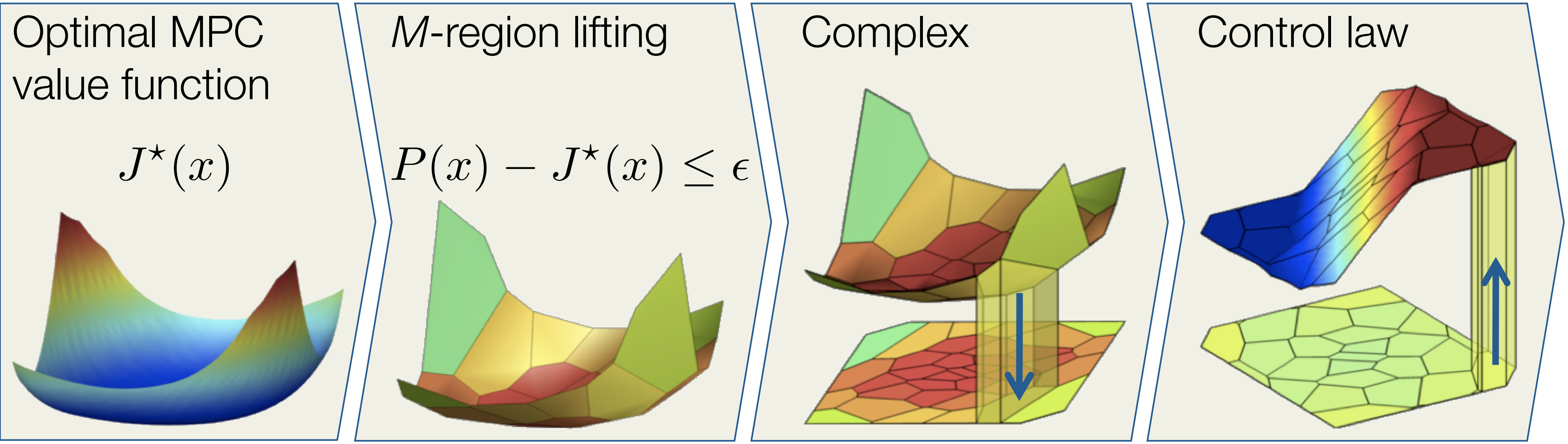
$$\tilde{u}(x) = \sum_{v} u^\star(v) \prod_{j=1}^{d} \max\left\{0,\ \frac{|x_j - v_j| + 1}{h}\right\}$$

- Defined on hierarchical grid
- High data storage
- *Evaluation in ns*



[Summers, Jones, Lygeros, Morari 2009]

# Real-time explicit MPC : Properties




Real-time explicit MPC:

- Is computable in micro- to nanoseconds <= Lifting function
- Satisfies constraints <= Barycentric interpolation
- **Stabilizes the system**
- Complexity/performance tradeoff

# $\varepsilon$-approx controller is stable if $\varepsilon < 1$

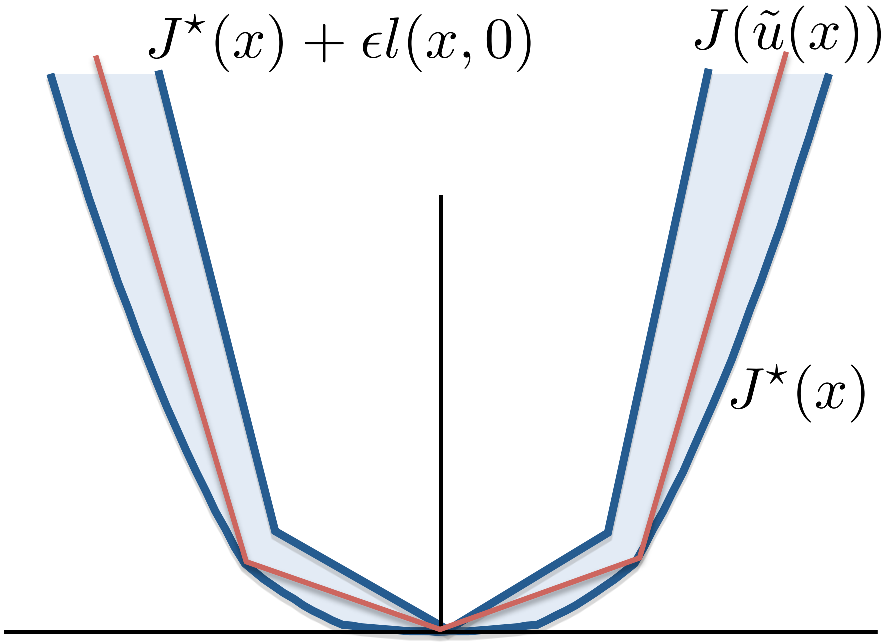
$$J(u) := V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\begin{aligned} J^\star(x_0) := \min_{u_i} \ & J(u) \\ \text{s.t. } & x_{i+1} = f(x_i, u_i) \\ & (x_i, u_i) \in \mathcal{X} \times \mathcal{U} \\ & x_N \in \mathcal{X}_N \end{aligned}$$

Thm: $x^+ = f(x, \tilde{u}(x))$
is stable if

$$J^\star(x) \leq J(\tilde{u}(x)) \leq J^\star(x) + \epsilon l(x, 0)$$

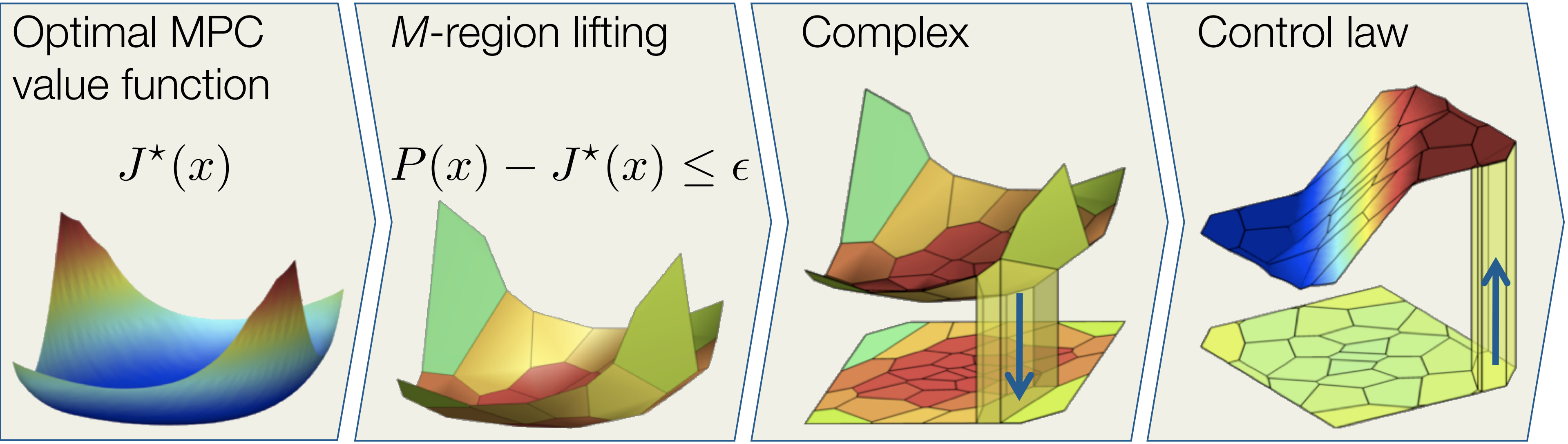
for $\epsilon < 1$




- Find a lifting use it to define $\tilde{u}(x)$
- Sufficiently close to optimal => Stabilizing

Convex optimization verifies stability condition
*without knowing optimal explicit solution*

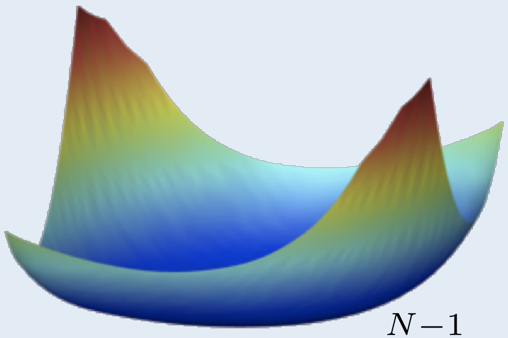
# Real-time explicit MPC : Properties




Real-time explicit MPC:

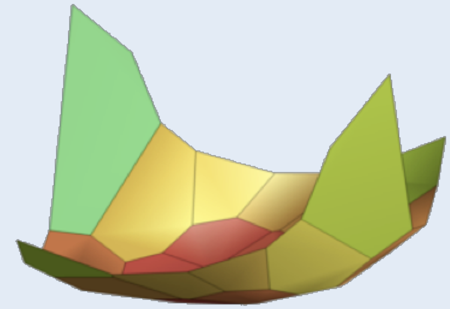
- Is computable in micro- to nanoseconds <= Lifting function
- Satisfies constraints <= Barycentric interpolation
- Stabilizes the system <= Error less than one
- **Complexity/performance tradeoff**

# Approximate Convex Parametric Programming
# *M*-region approximation => Double description method



$$J^\star(x_0) = \min_{u_i} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\begin{aligned} \text{s.t. } x_{i+1} &= f(x_i, u_i) \\ (x_i, u_i) &\in \mathcal{X} \times \mathcal{U} \\ x_N &\in \mathcal{X}_N \end{aligned}$$

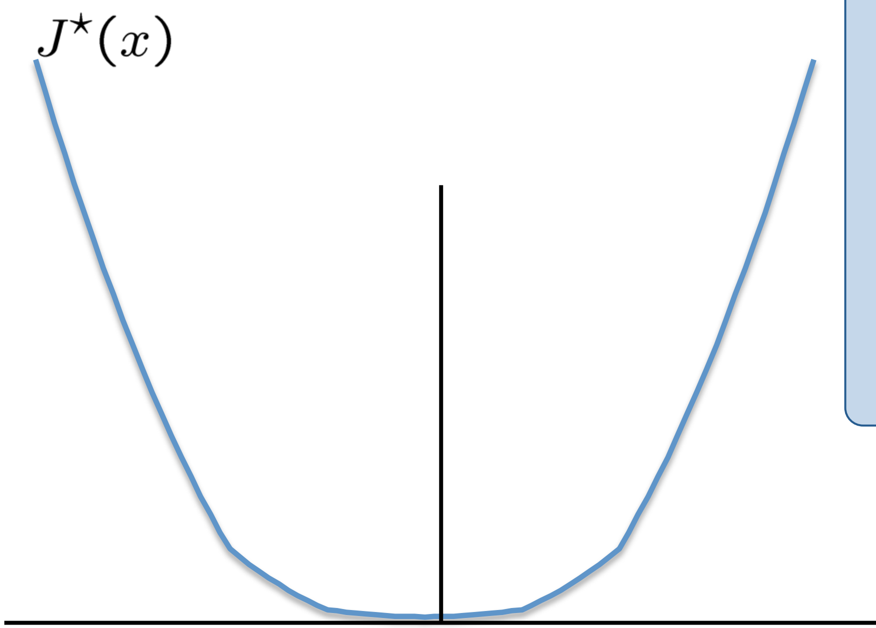


$$P(x)$$

- Open problem in many areas:
  - Vertex enumeration, Projection, Non-negative matrix factorization…
  - These problems are known to be NP-hard
- Poly-time greedy-optimal algorithm

$\Rightarrow$ Lifting of *M* regions => Iterate algorithm *M* times!

# Polyhedral approximation



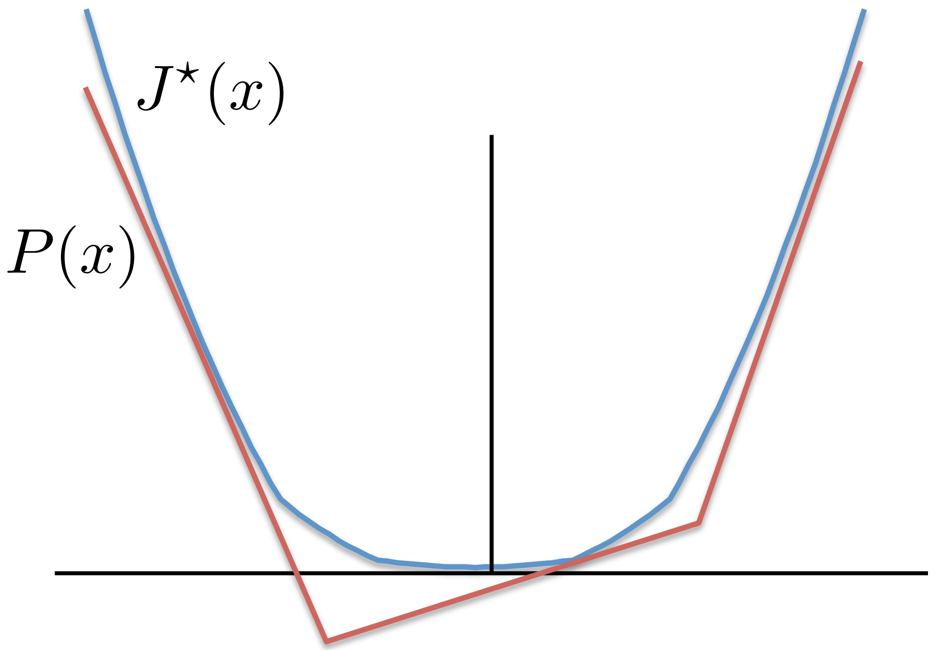

$$J^\star(x_0) = \min_{u_i} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$$
$$x_N \in \mathcal{X}_N$$

$\text{epi}\, J^\star(x)$

- Implicitly defined
- Convex set
- Lyapunov function

Goal: Greedy-optimal polyhedral approx. of *M*-facets

# Polyhedral approximation

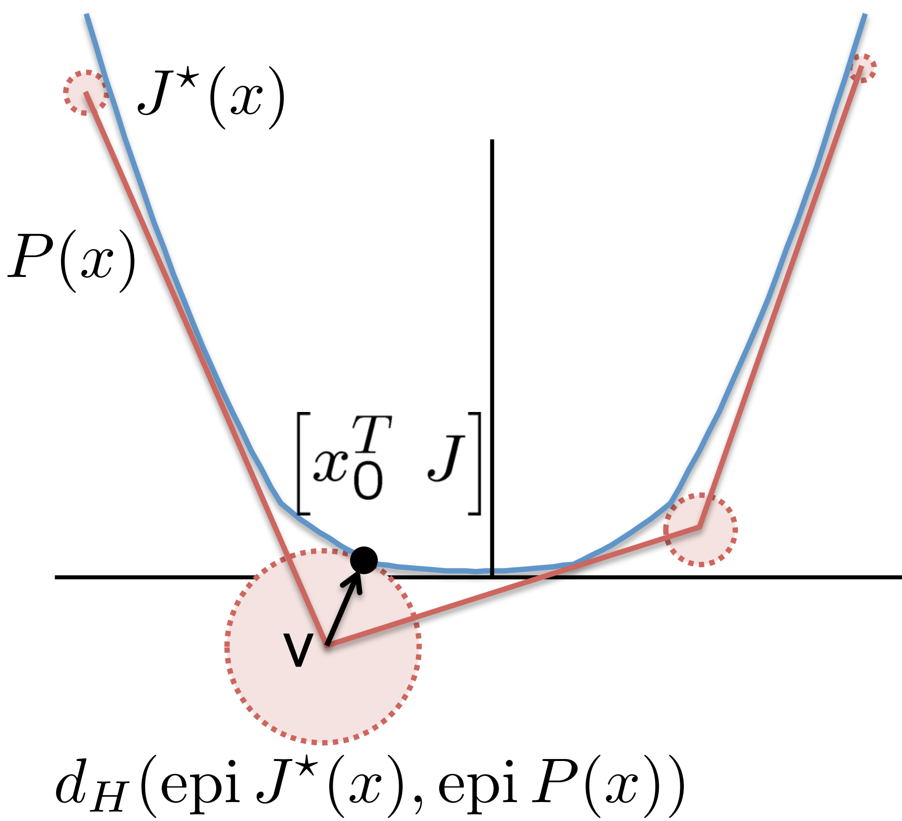



1. Outer polyhedral approx

Compute Hausdorff distance

$$d_H(\text{epi}\, J^\star(x), \text{epi}\, P(x)) = \max_{v \in \text{epi}\, P(x)} \min_{y \in \text{epi}\, J^\star(x)} \|v - y\|$$

# Polyhedral approximation

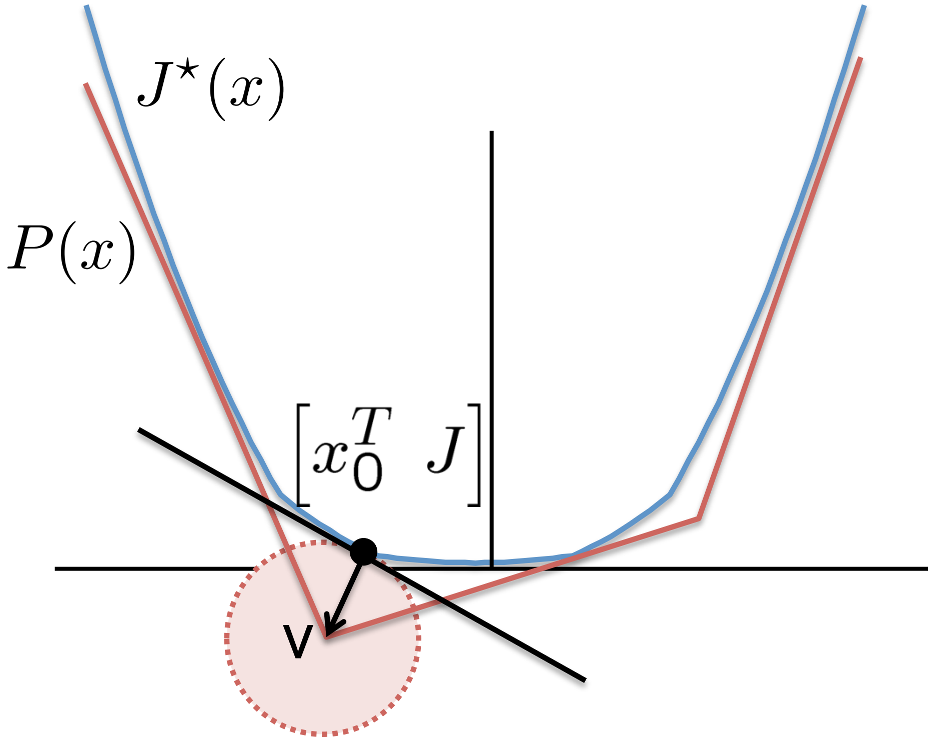



1. Outer polyhedral approx
2. Project vertices onto optimal cost

$$
\begin{aligned}
\min \quad & \left\| v - \begin{bmatrix} x_0^T & J \end{bmatrix}^T \right\|_2^2 \\
\text{s.t.} \quad & J \geq V_N(x_N) + \sum_{k=0}^{N-1} l(x_i, u_i) \\
& x_{i+1} = f(x_i, u_i) \\
& (x_i, u_i) \in \mathcal{X} \times \mathcal{U}
\end{aligned}
$$

- Convex optimization!
- Do not need to know optimal solution

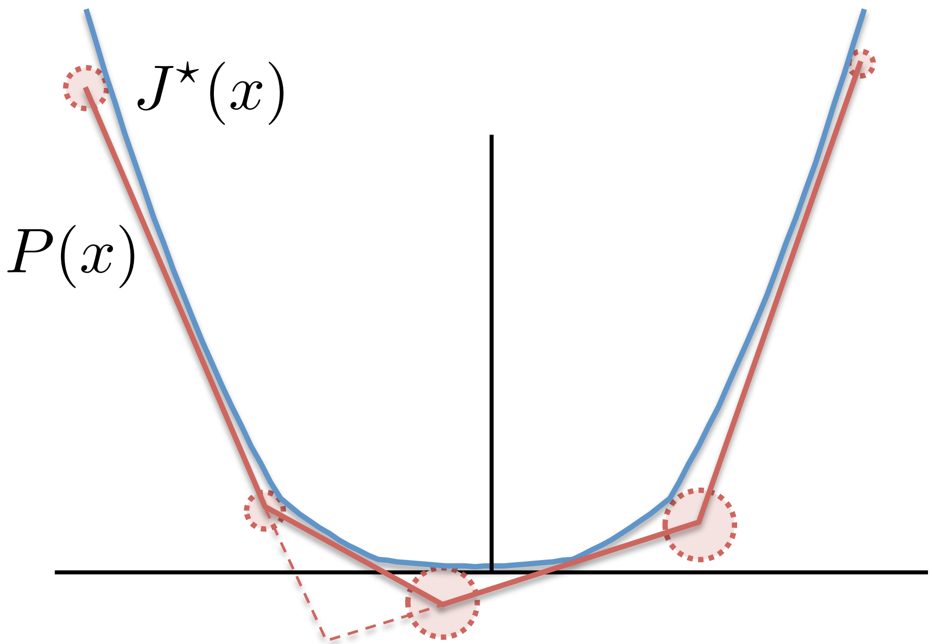
# Polyhedral approximation




1. Outer polyhedral approx
2. Project vertices onto optimal cost
3. Insert maximally separating hyperplane

Maximally reduce Hausdorff distance

# Polyhedral approximation

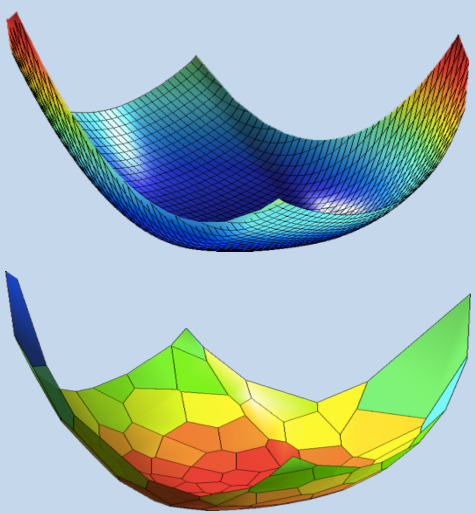



1. Outer polyhedral approx
2. Project vertices onto optimal cost
3. Insert maximally separating hyperplane
4. Repeat

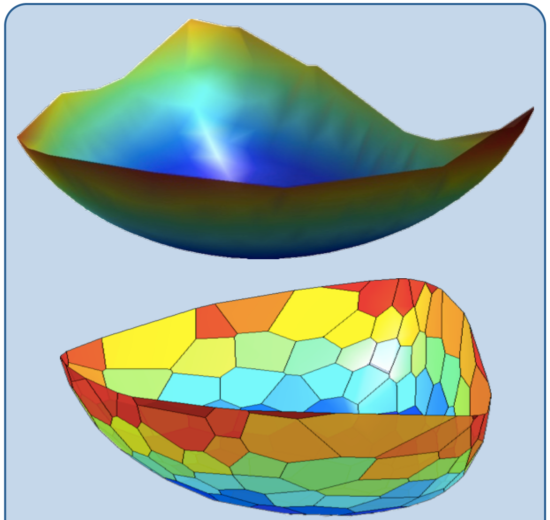
Result: M-region approximation with greedy-minimal Hausdorff distance.

# Lifting calculation : Algorithm properties

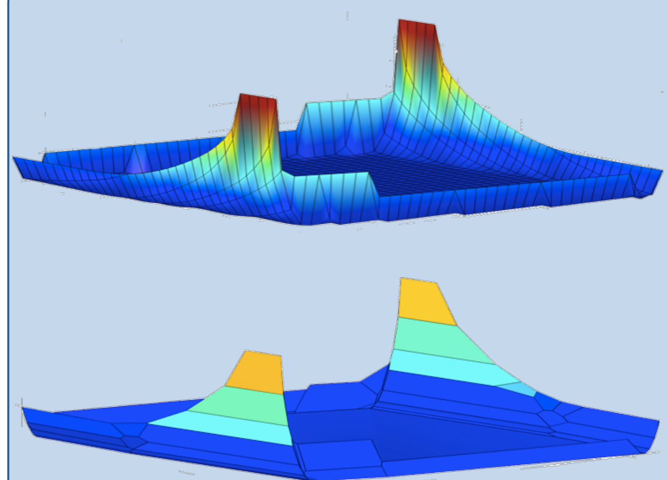
- Lifting of *M* regions <= Iterate algorithm *M* times
- Monotonic decrease in Hausdorff distance
  - Complexity / performance tradeoff via *M*
- There exists a minimum *M* for stability
  - $\varepsilon$-error in finite time $\Rightarrow$ will find a Lyapunov function
  - Once stable, always stable



Quadratic programming

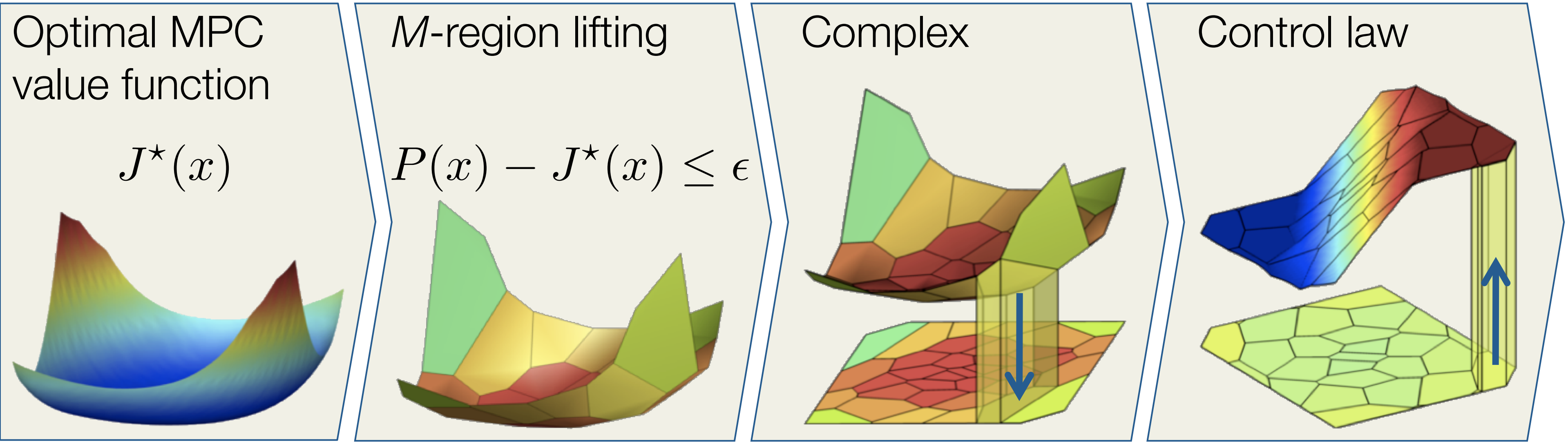


Geometric programming



Second-order cone programming

# Real-time explicit MPC : Properties




Real-time explicit MPC:

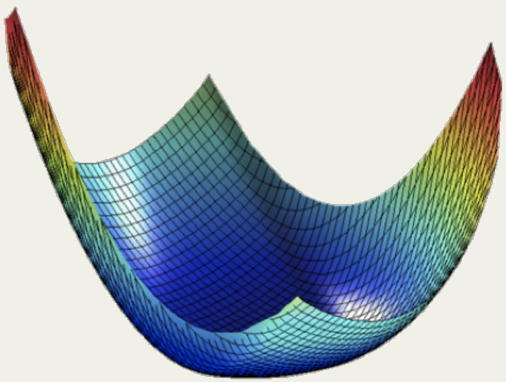
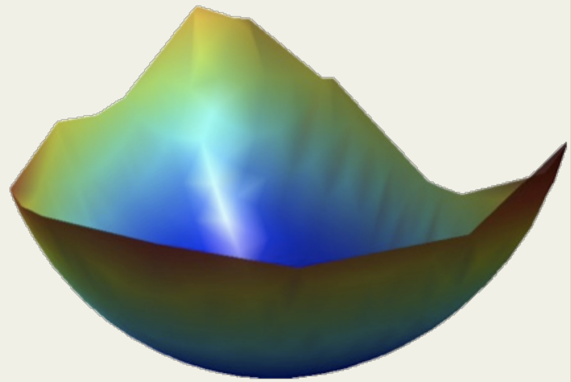
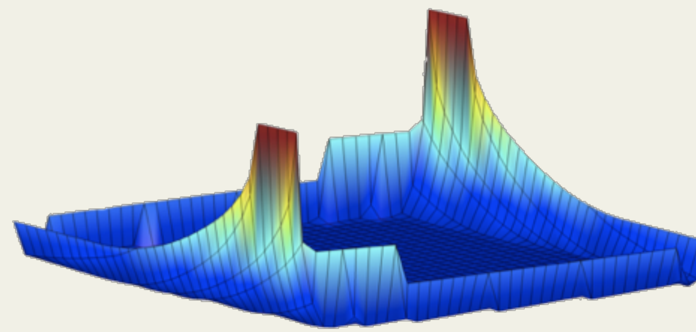
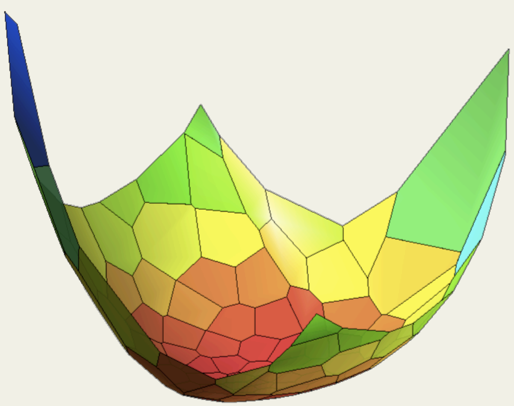
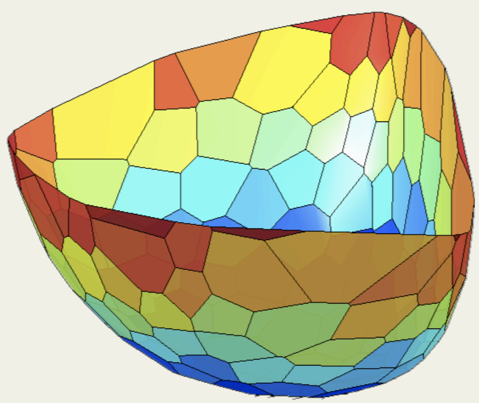
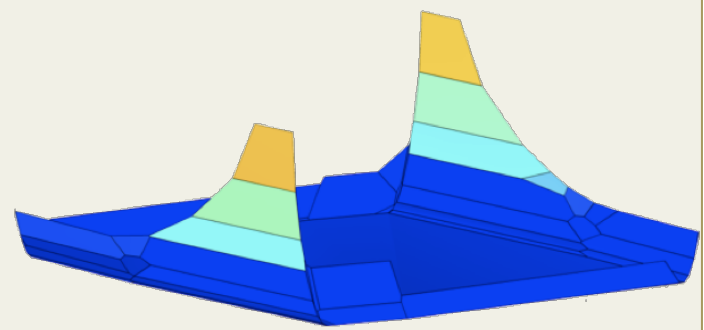
| | |
|---|---|
| Is computable in micro- to nanoseconds | <= Lifting function |
| Satisfies constraints | <= Barycentric interpolation |
| Stabilizes the system | <= Error less than one |
| Complexity/performance tradeoff | <= *M*-region lifting |

***Verified*** **stability, feasibility & time *without knowing optimal solution***



*[C.N. Jones and M. Morari, IEEE-TAC 2010]*

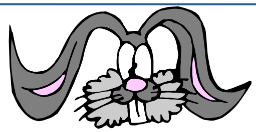
# Double description method
# Applicable to all convex parametric / set operations

| Quadratic programming | Geometric programming | Second-order cone programming |
|---|---|---|
|  |  |  |
|  |  |  |

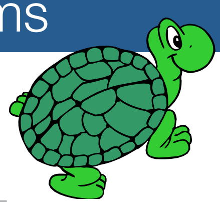


[C.N. Jones and M. Morari, IEEE-TAC 2010]

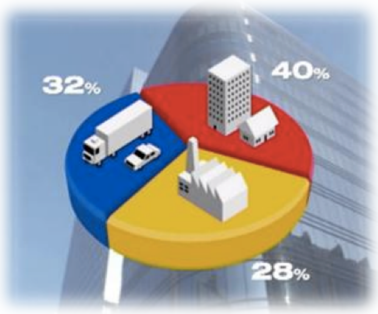
# Applications by the Automatic Control Lab



| | | |
|---|---|---|
| 18 ns | Multi-core thermal management (EPFL) | *[Zanini et al 2010]* |
| 10 µs | Voltage source inverters | *[Mariethoz et al 2008]* |
| 20 µs | DC/DC converters (STM) | *[Mariethoz et al 2008]* |
| 25 µs | Direct torque control (ABB) | *[Papafotiou 2007]* |
| 50 µs | AC / DC converters | *[Richter et al 2010]* |
| 5 ms | Electronic throttle control (Ford) | *[Vasak et al 2006]* |
| 20 ms | Traction control (Ford) | *[Borrelli et al 2001]* |
| 40 ms | Micro-scale race cars | |
| 50 ms | Autonomous vehicle steering (Ford) | *[Besselmann et al 2008]* |
| 500 ms | Energy efficient building control (Siemens) | *[Oldewurtel et al 2010]* |



# Example: Energy Efficient Building Climate Control




Building sector: 40% of energy worldwide!
*[International Energy Agency 2008]*

Idea: Incorporate weather predictions into building controller

*"The trouble with weather forecasting is that it's right too often for us to ignore it and wrong too often for us to rely on it."*

*- Patrick Young*

SIEMENS Building Technologies

MeteoSwiss Schweizerische Eidgenossenschaft Confédération suisse Confederazione Svizzera Confederaziun svizra

EMPA Materials Science & Technology

swisselectric research

CCEM-CH Competence Center Energy and Mobility

EPFL AUTOMATIC CONTROL LABORATORY

# Example: Energy Efficient Building Climate Control

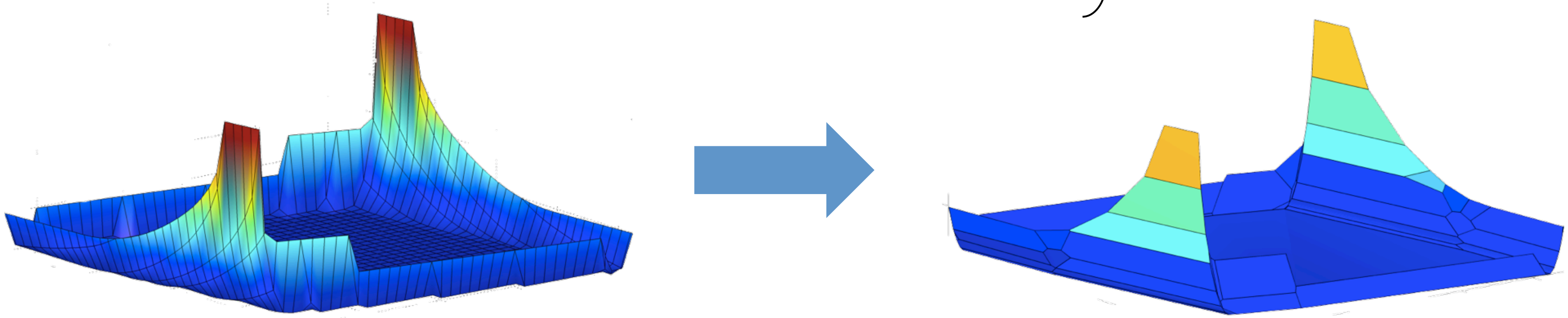
- Analysis shows uncertainty in weather forecast » Gaussian
- Stochastic model predictive controller
  - Constraints violated with a specified probability

$$\min \mathbb{E}\left[\sum_{k=0}^{N-1} c^T \cdot \mu_k(\phi_k(x_0, \mu, \mathbf{w}))\right]$$

$$\text{s.t. } \mu_k(\phi_k(x_0, \mu, \mathbf{w})) \in \mathcal{U} \quad \forall \mathbf{w} \in \mathbb{R}^{mN}$$

$$\mathbb{P}\{\phi_k(x_0, \mu, \mathbf{w}) \in \mathcal{X}\} \geq 1 - \alpha$$

} Convex parametric second-order cone problem

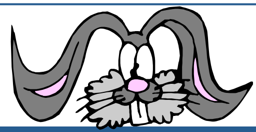


**PWA controller with 30 regions => Can run in a light-switch**
**Energy savings in idealized Swiss buildings from 5% to 40%**

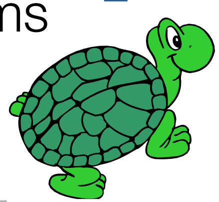


*[F. Oldewurtel, A. Parisio, C.N. Jones and M. Morari 2009]*

# Applications by the Automatic Control Lab

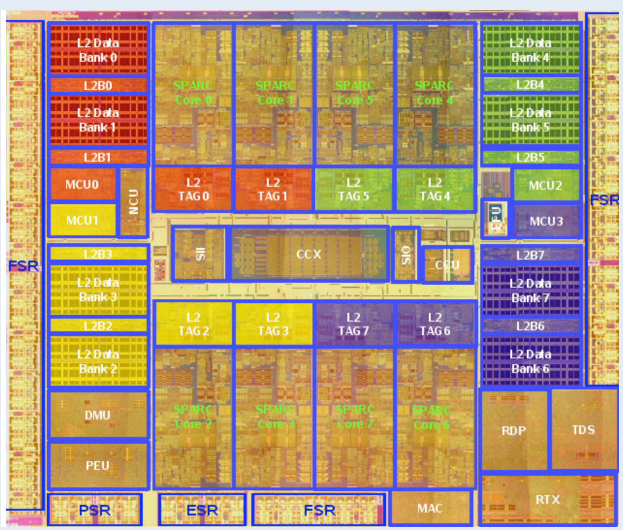


| | | |
|---|---|---|
| 18 ns | Multi-core thermal management (EPFL) | *[Zanini et al 2010]* |
| 10 μs | Voltage source inverters | *[Mariethoz et al 2008]* |
| 20 μs | DC/DC converters (STM) | *[Mariethoz et al 2008]* |
| 25 μs | Direct torque control (ABB) | *[Papafotiou 2007]* |
| 50 μs | AC / DC converters | *[Richter et al 2010]* |
| 5 ms | Electronic throttle control (Ford) | *[Vasak et al 2006]* |
| 20 ms | Traction control (Ford) | *[Borrelli et al 2001]* |
| 40 ms | Micro-scale race cars | |
| 50 ms | Autonomous vehicle steering (Ford) | *[Besselmann et al 2008]* |
| 500 ms | Energy efficient building control (Siemens) | *[Oldewurtel et al 2010]* |



# Example :
# Temperature Regulation of Multi-Core Processor

- Goals
  - Track workload requests
  - Minimize power usage
  - Respect temperature limits
- Quadratic nonlinear dynamics
  - Exact convex relaxation
- Stringent computational and storage requirements




$$J^\star(x_0, w) = \min_{f_i} \sum_{t=0}^{N} \sum_{i=0}^{t} (w_i - f_i)$$

$$\text{s.t. } x_{i+1} = Ax_i + Bf_i^2$$

$$\sum_{i=0}^{t} w_i \leq \sum_{i=0}^{t} f_i$$

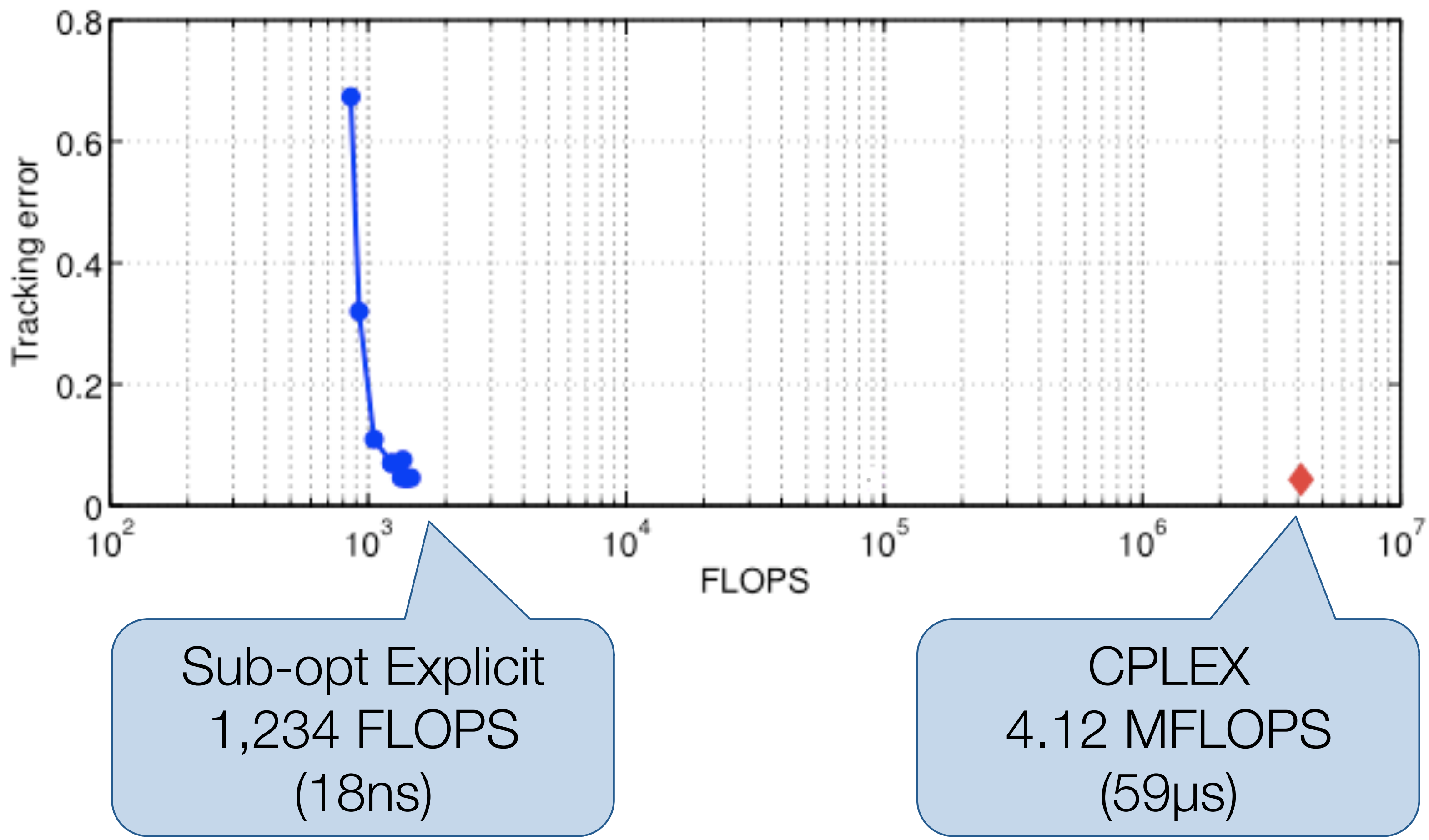
$$x_i \leq T_{\max}$$

$$f_{\min} \leq f_i \leq f_{\max}$$



*[F. Zanini, C.N. Jones, D. Atienza, and G. De Micheli, 2010]*

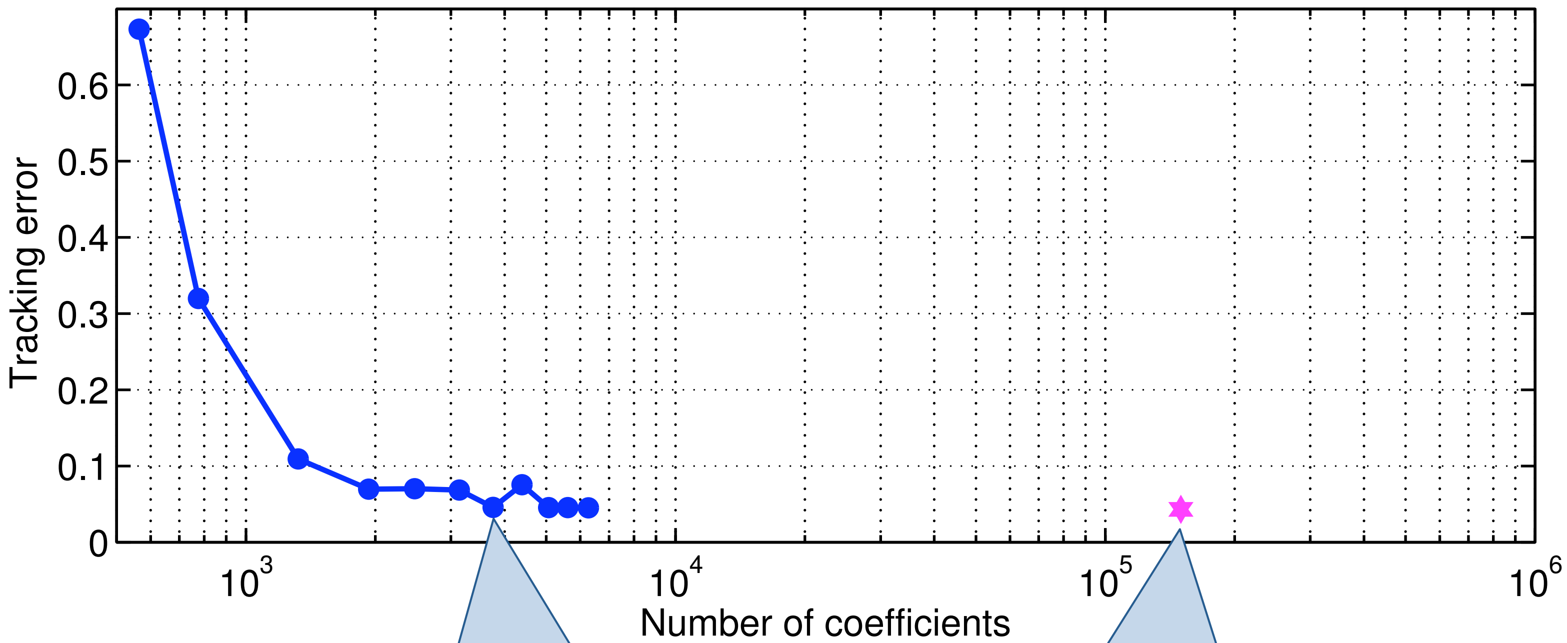
# Computational results for QCQP : >3,000× faster




(Assuming 70 GFLOPS/sec – e.g., Intel Core i7 965 XE)

**>3,000× faster than CPLEX**

# Computational results for QCQP : 45× less storage



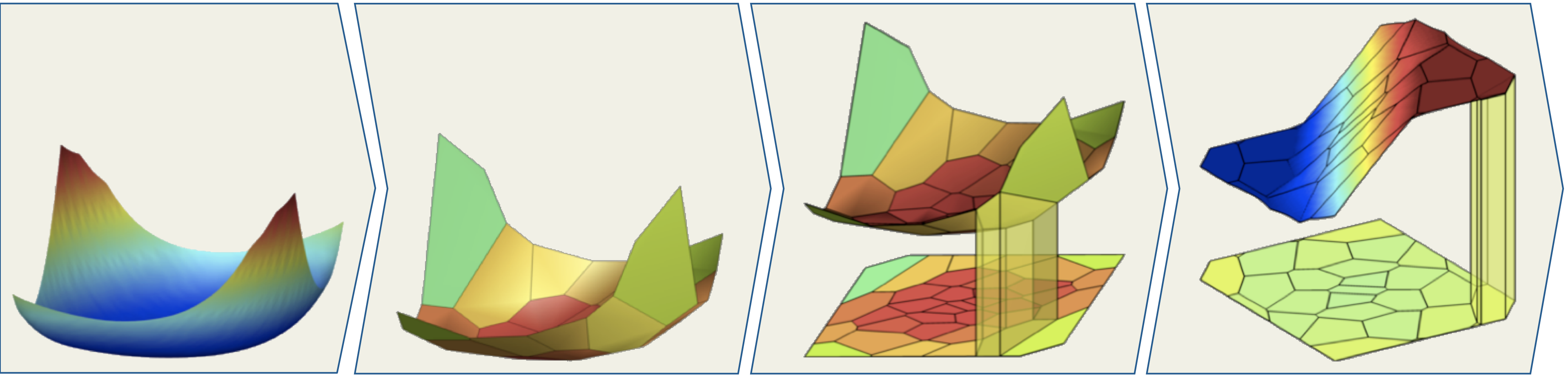

Sub-opt Explicit
26 KB

Linearised explicit
1.14 MB

**45× less storage**

# Summary

- Complexity of explicit MPC control laws highly variable
  - Depends on tuning, dynamics, problem size, etc
- Fixed-complexity MPC
  - Produce sub-optimal control law of *specified* complexity
  - Certificate of stability, invariance
- Several related approximate explicit MPC methods
  - Different pros/cons
  - Based on sampling optimal control law and interpolating



# Workshop Outline

| | | |
|---|---|---|
| 14:00 – 14:30 | Introduction | |
| 14:30 – 15:15 | Lecture 1: | MPC Theory |
| 15:15 – 15:20 | Break | |
| 15:20 – 15:40 | Lecture 2 : | Explicit MPC |
| 15:40 – 16:00 | Break | |
| 16:00 – 16:40 | Lecture 2 : | Explicit MPC |
| 16:40 – 17:10 | Lecture 3 : | Real-time explicit MPC |
| 17:10 – 17:20 | Break | |
| 17:20 – 18:00 | Lecture 4 : | Real-time online optimization |